INFLATION INERTIA AND CREDIBLE DISINFLATION

THE OPEN ECONOMY CASE

ADDITIONAL APPENDIX: COMPLETE DERIVATIONS

(Not for Publication)

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1 Firms

1.1 Maximization Problem

Firms maximize the present discounted value of real future profits in terms of nontradables:

$$\underset{V_t^j, v_t^j}{Max} \quad \int_t^{\infty} e^{-\int_t^s (\delta + r + \varepsilon_r - \pi_r) dr} \left[\frac{V_t^j e^{v_t^j (s-t)}}{P_s} y_s(j)(1 + sub) - w_s l_s(j) \right] ds \text{ , s.t.}$$

$$y_s(j) = c_s \left(\frac{V_t^j e^{v_t^j (s-t)}}{P_s} \right)^{-\sigma} ,$$

$$y_s(j) = l_s(j) \ .$$

- Nominal revenue at $s \geq t$: $V_t^j e^{v_t^j (s-t)} y_s(j)(1 + sub)$.
- Nominal cost at $s \geq t$: $W_s l_s(j)$.
- Divided by nontradables price level P_s.
- Discounted at own rate of interest for nontradables $r + \varepsilon_t - \pi_t$.
- Weighted by the probability that today's price will still be in force.
- Can be rewritten as:

$$\underset{V_t^j, v_t^j}{Max} \quad \int_t^{\infty} e^{-\int_t^s (\delta + r + \varepsilon_r - \pi_r) dr} \left[\left(\frac{V_t^j e^{v_t^j (s-t)}}{P_s} (1 + sub) - w_s \right) \left(c_s \left(\frac{V_t^j e^{v_t^j (s-t)}}{P_s} \right)^{-\sigma} \right) \right] ds \ . \tag{1}$$

1.2 First-Order Conditions

For V_t

- Take FOC, then multiply through by V_t to get:

$$\int_t^\infty \left[e^{-\int_t^s(\delta+r+\varepsilon_r-\pi_r)dr} c_s \left(\frac{V_t e^{v_t(s-t)}}{P_s}\right)^{-\sigma} \frac{V_t e^{v_t(s-t)}}{P_s}(1+sub) \right.$$
$$\left. -\sigma c_s \left(\frac{V_t e^{v_t(s-t)}}{P_s}\right)^{-\sigma} \left(\frac{V_t e^{v_t(s-t)}}{P_s}(1+sub) - w_s\right)\right] ds = 0 \ .$$

- Rearrange:

$$\int_t^\infty e^{-\int_t^s(\delta+r+\varepsilon_r-\pi_r)dr} c_s \left(\frac{V_t e^{v_t(s-t)}}{P_s}\right)^{-\sigma} \left(\frac{V_t e^{v_t(s-t)}}{P_s}(1+sub)\frac{\sigma-1}{\sigma} - w_s\right) ds = 0 \ .$$

- Use $sub = (\sigma-1)^{-1}$, define $p_t \equiv V_t/P_t$, use $P_s = P_t e^{\int_t^s \pi_r dr}$ for $s \geq t$, to get:

$$\int_t^\infty e^{-\int_t^s(\delta+r+\varepsilon_r-\pi_r)dr} c_s \left(p_t e^{-\int_t^s(\pi_r-v_t)dr}\right)^{-\sigma} \left[p_t e^{-\int_t^s(\pi_r-v_t)dr} - w_s\right] ds = 0 \quad . \qquad (2)$$

For v_t

- Same steps as above lead to:

$$\int_t^\infty e^{-\int_t^s(\delta+r+\varepsilon_r-\pi_r)dr} c_s \left(p_t e^{-\int_t^s(\pi_r-v_t)dr}\right)^{-\sigma} (s-t) \left[p_t e^{-\int_t^s(\pi_r-v_t)dr} - w_s\right] ds = 0 \quad . \ (3)$$

1.3 Linearization

For V_t

- Steady state: $\bar{\pi} = \bar{v} = \bar{\varepsilon}$, $\bar{p} = 1$, $\bar{w} = 1$.
- FOC (2) can be rewritten as:

$$\begin{aligned} &\int_t^\infty e^{-\int_t^s (\delta + r + \varepsilon_r - \pi_r)dr} c_s \left(p_t e^{-\int_t^s (\pi_r - v_t)dr} \right)^{1-\sigma} ds \\ = &\int_t^\infty e^{-\int_t^s (\delta + r + \varepsilon_r - \pi_r)dr} c_s \left(p_t e^{-\int_t^s (\pi_r - v_t)dr} \right)^{-\sigma} w_s ds = 0 \quad . \end{aligned}$$

- Linearized terms pertaining to the discount factor $e^{-\int_t^s (\delta + r + \varepsilon_r - \pi_r)dr}$ and all $\hat{c}_s$ terms are equal on the left- and right hand side and therefore drop out.
- Remaining terms under the integral on the left-hand side are:

$$e^{-(\delta + r)(s-t)} \bar{c}(1 - \sigma) \left[\hat{p}_t - \int_t^s (\hat{\pi}_r - \hat{v}_t)dr \right] \; .$$

- Remaining terms under the integral on the right-hand side are:

$$e^{-(\delta + r)(s-t)} \bar{c}(-\sigma) \left[\hat{p}_t - \int_t^s (\pi_r - \hat{v}_t)dr \right] + e^{-(\delta + r)(s-t)} \bar{c} \hat{w}_s \quad .$$

- We subtract the right-hand side from the left-hand side, divide through by $\bar{c}$, and get:

$$0 = \int_t^\infty e^{-(\delta + r)(s-t)} \left[\hat{p}_t - \hat{w}_s - \int_t^s (\pi_r - \hat{v}_t)dr \right] ds \quad . \tag{4}$$

- Next we use the following two properties of exponential distributions:

$$\delta \int_t^\infty (s - t)e^{-\delta(s-t)} ds = \frac{1}{\delta} \quad , \quad \delta \int_t^\infty (s - t)^2 e^{-\delta(s-t)} ds = \frac{2}{\delta^2}.$$

- Use these to simplify the terms involving $\hat{p}_t$ and $\hat{v}_t$:

$$\int_t^\infty e^{-(\delta + r)(s-t)} \left[\hat{p}_t + (s - t)\hat{v}_t \right] ds = \int_t^\infty e^{-(\delta + r)(s-t)} \left[\int_t^s \hat{\pi}_r dr + \hat{w}_s \right] ds \quad .$$

$$\frac{(p_t - 1)}{(\delta + r)} + \frac{(v_t - \bar{\varepsilon})}{(\delta + r)^2} = \int_t^\infty e^{-(\delta + r)(s-t)} \left[\int_t^s \pi_r dr - (s - t)\bar{\varepsilon} + w_s - 1 \right] ds = 0 \quad .$$

$$\frac{p_t}{(\delta + r)} + \frac{v_t}{(\delta + r)^2} = \int_t^\infty e^{-(\delta + r)(s-t)} \left[\int_t^s \pi_r dr + w_s \right] ds = 0 \quad .$$

- Linearized FOC:

$$p_t + \frac{v_t}{\delta + r} = (\delta + r) \int_t^\infty e^{-(\delta + r)(s-t)} \left[\int_t^s \pi_r dr + w_s \right] ds = 0 \quad . \tag{5}$$

For v_t

- The FOC is the same as for V_t except for the additional factor $(s-t)$. By following the same steps as above, we therefore get the following expression:

$$0 = \int_t^\infty e^{-(\delta+r)(s-t)}(s-t)\left[\hat{p}_t - \hat{w}_s - \int_t^s (\pi_r - \hat{v}_t)dr\right] ds \quad . \tag{6}$$

- Use the same steps as above, including the second above mentioned property of exponential distributions, to get:

$$\frac{p_t}{\delta+r} + \frac{2v_t}{(\delta+r)^2} = (\delta+r)\int_t^\infty e^{-(\delta+r)(s-t)}(s-t)\left[\int_t^s \pi_r dr + w_s\right] ds = 0 \quad . \tag{7}$$

1.4 Time Derivatives of Linearized FOC

Using Leibnitz's rule we can differentiate (5) and (7).

For V_t

- Leibnitz's rule:

$$\dot{p}_t + \frac{\dot{v}_t}{\delta + r} = -(\delta + r)w_t + (\delta + r)(p_t + \frac{v_t}{\delta + r}) - \pi_t(\delta + r)\int_t^\infty e^{-(\delta+r)(s-t)}ds \quad .$$

- Simplify:

$$\dot{p}_t + \frac{\dot{v}_t}{\delta + r} = (\delta + r)(p_t - w_t) + v_t - \pi_t \quad . \tag{8}$$

For v_t

- Leibnitz's rule:

$$\frac{\dot{p}_t}{(\delta + r)} + \frac{2\dot{v}_t}{(\delta + r)^2} = -(\delta + r)0 + (\delta + r)\left(\frac{p_t}{\delta + r} + \frac{2v_t}{(\delta + r)^2}\right)$$
$$-(\delta + r)\int_t^\infty e^{-(\delta+r)(s-t)}\left[\int_t^s \pi_r dr + w_s\right]ds - \pi_t(\delta + r)\int_t^\infty e^{-(\delta+r)(s-t)}ds \; .$$

- The second to last term can be replaced with $(p_t + v_t/(\delta + r))$ from (5). We get:

$$\frac{\dot{p}_t}{(\delta + r)} + \frac{2\dot{v}_t}{(\delta + r)^2} = \left(p_t + \frac{2v_t}{(\delta + r)}\right) - \left(p_t + \frac{v_t}{(\delta + r)}\right) - \frac{\pi_t}{(\delta + r)} \; .$$
$$\frac{\dot{p}_t}{(\delta + r)} + \frac{2\dot{v}_t}{(\delta + r)^2} = \frac{v_t - \pi_t}{(\delta + r)} \; .$$
$$\dot{p}_t + \frac{2\dot{v}_t}{\delta + r} = v_t - \pi_t \quad . \tag{9}$$

- Combine (8) and (9):

$$\dot{v}_t = -(\delta + r)^2 (p_t - w_t) \quad . \tag{10}$$

2 The Aggregate Price Index

2.1 Formula for the Index

$$P_t = \left(\delta \int_{-\infty}^{t} e^{-\delta(t-s)} (V_s e^{v_s(t-s)})^{1-\sigma} ds \right)^{\frac{1}{1-\sigma}} \quad . \tag{11}$$

2.2 Linearize P_t Itself

- Divide by $P_t^{1-\sigma}$, then use $p_s \equiv V_s/P_s$, use $P_t = P_s e^{\int_s^t \pi_r dr}$ for $s \leq t$:

$$1 = \int_{-\infty}^{t} \delta e^{-\delta(t-s)} (1-\sigma) \left(p_s e^{-\int_s^t (\pi_r - v_s) dr} \right)^{1-\sigma} ds \ .$$

- Then linearize:

$$\int_{-\infty}^{t} \delta e^{-\delta(t-s)} (1-\sigma) \left(\hat{p}_s - \int_s^t (\hat{\pi}_r - \hat{v}_s) dr \right) ds = 0 \ . \tag{12}$$

2.3 First Time Derivative of P_t

- Differentiate (11):

$$\dot{P}_t = \frac{1}{1-\sigma} P_t^{\sigma} \left[\delta V_t^{1-\sigma} - \delta P_t^{1-\sigma} + \int_{-\infty}^{t} \delta e^{-\delta(t-s)} (1-\sigma) (V_s e^{v_s(t-s)})^{1-\sigma} v_s ds \right] \ .$$

$$\frac{\dot{P}_t}{P_t} = \frac{1}{1-\sigma} P_t^{-(1-\sigma)} \left[\delta V_t^{1-\sigma} - \delta P_t^{1-\sigma} + \int_{-\infty}^{t} \delta e^{-\delta(t-s)} (1-\sigma) (V_s e^{v_s(t-s)})^{1-\sigma} v_s ds \right] \ .$$

- Simplify and use $\pi_t = \dot{P}_t / P_t$:

$$\pi_t = \frac{1}{1-\sigma} \left[\delta p_t^{1-\sigma} - \delta + \delta(1-\sigma) \int_{-\infty}^{t} e^{-\delta(t-s)} \left(p_s e^{-\int_s^t (\pi_r - v_s) dr} \right)^{1-\sigma} v_s ds \right] \ .$$

$$\pi_t = \frac{\delta}{1-\sigma} \left(p_t^{1-\sigma} - 1 \right) + \delta \int_{-\infty}^{t} e^{-\delta(t-s)} \left(p_s e^{-\int_s^t (\pi_r - v_s) dr} \right)^{1-\sigma} v_s ds \ .$$

2.4 Linearize the First Time Derivative

- Linearize:

$$\hat{\pi}_t = \delta \hat{p}_t + \int_{-\infty}^{t} \delta e^{-\delta(t-s)} \left[\hat{v}_s + \bar{\varepsilon}(1-\sigma) \left(\hat{p}_s - \int_s^t (\hat{\pi}_r - \hat{v}_s) dr \right) \right] ds \ .$$

- The final part of this expression equals zero by (12), so we get:

$$\hat{\pi}_t = \delta \hat{p}_t + \int_{-\infty}^{t} \delta e^{-\delta(t-s)} \hat{v}_s ds \ .$$

$$(\pi_t - \bar{\varepsilon}) = \delta(p_t - 1) + \int_{-\infty}^{t} \delta e^{-\delta(t-s)} \left(v_s - \bar{\varepsilon} \right) ds \ . \tag{13}$$

2.5 The Auxiliary Variable

- We denote the final expression in (13) as ψ_t:

$$\psi_t = \int_{-\infty}^{t} \delta e^{-\delta(t-s)} v_s ds \ . \tag{14}$$

- Leibnitz's rule gives us:

$$\dot{\psi}_t = \delta v_t - \delta \psi_t \ .$$

- Or equivalently:

$$\dot{\psi}_t = \delta \left(v_t - \bar{\varepsilon} \right) - \delta \left(\psi_t - \bar{\varepsilon} \right) \ . \tag{15}$$

- Use (13) and (14) to obtain the following equation, which will be useful below:

$$\hat{p}_t = \frac{1}{\delta} \left(\hat{\pi}_t - \hat{\psi}_t \right) \ . \tag{16}$$

2.6 Time Derivative of the Linearized Inflation Equation

- Differentiate (13) using Leibnitz's rule:

$$\dot{\pi}_t = \delta\dot{p}_t + \delta v_t - \delta(\pi_t - \delta(p_t - 1)) \; .$$
$$\dot{\pi}_t = \delta\dot{p}_t + \delta(v_t - \pi_t) + \delta^2(p_t - 1)) \; .$$

- To substitute out $\dot{p}_t$, use (9) and (10):

$$\dot{p}_t + \frac{2\dot{v}_t}{\delta + r} = v_t - \pi_t \; .$$
$$\dot{v}_t = -(\delta + r)^2 \left(p_t - w_t\right) \; .$$

- Combine these to get:

$$\dot{p}_t = (v_t - \pi_t) + 2(\delta + r)\left(p_t - w_t\right) \; .$$

- Now substitute out $\dot{p}_t$:

$$\dot{\pi}_t = \delta\left(v_t - \pi_t\right) + 2\delta(\delta + r)\left(p_t - w_t\right) + \delta(v_t - \pi_t) + \delta^2(p_t - 1)) \; .$$

- Or equivalently:

$$\dot{\pi}_t = 2\delta\left(\hat{v}_t - \hat{\pi}_t\right) + 2\delta(\delta + r)\left(\hat{p}_t - \hat{w}_t\right) + \delta^2\hat{p}_t \; .$$
$$\dot{\pi}_t = 2\delta\left(\hat{v}_t - \hat{\pi}_t\right) + (3\delta^2 + 2\delta r)\hat{p}_t - 2\delta(\delta + r)\hat{w}_t \; .$$

- Use (16) to substitute out $\hat{p}_t$:

$$\dot{\pi}_t = 2\delta\hat{v}_t - 2\delta\hat{\pi}_t + (3\delta + 2r)\hat{\pi}_t - (3\delta + 2r)\hat{\psi}_t - 2\delta(\delta + r)\hat{w}_t \; .$$

$$\dot{\pi}_t = 2\delta\left(v_t - \bar{\varepsilon}\right) + (\delta + 2r)\left(\pi_t - \bar{\varepsilon}\right) - (3\delta + 2r)\left(\psi_t - \bar{\varepsilon}\right) - 2\delta(\delta + r)\left(w_t - 1\right) \; . \tag{17}$$

2.7 Final Expression for $\dot{v}_t$

- Use (16) again in (10) to get:

$$\dot{v}_t = \frac{(\delta + r)^2}{\delta}(\psi_t - \bar{\varepsilon}) - \frac{(\delta + r)^2}{\delta}(\pi_t - \bar{\varepsilon}) + (\delta + r)^2\left(w_t - 1\right) \; . \tag{18}$$

3 Marginal Cost Differential Equation

3.1 FOC

- From equation (11) in the paper we have:

$$w_t = \frac{\kappa c_t(1+\alpha i_t)}{(1-L_t)(1-\gamma)} \ .$$

3.2 Linearization

- Let $\hat{c}_t = \ln(c_t) - \ln(\bar{c})$, and note from Appendix A of the paper that $\hat{L}_t = \hat{c}_t$ and $\bar{L} = \bar{c}$. Then we have:

$$\hat{w}_t = \frac{1}{1-\bar{c}}\hat{c}_t + \frac{\alpha}{1+\alpha\bar{\imath}}\hat{\varepsilon}_t \ . \tag{19}$$

3.3 Differentiation

- Let $\hat{c}^*_t = \ln(c^*_t) - \ln(\bar{c}^*)$ and $\hat{e}_t = \ln(e_t) - \ln(\bar{e})$. Then the first-order condition (10) in the paper implies that $\hat{c}_t = \hat{c}^*_t + \hat{e}_t$. This in turn implies the following for the time derivatives:

$$\frac{\dot{c}_t}{c_t} = \frac{\dot{c}^*_t}{c^*_t} + \frac{\dot{e}_t}{e_t} \ .$$

- Under our assumptions about monetary policy, first-order condition (9) in the paper implies that $\dot{\varepsilon}_t = 0$ and therefore $\dot{c}^*_t / c^*_t = 0$. (This is modified appropriately for the computations underlying the welfare analysis.)
- Furthermore, we have

$$\frac{\dot{e}_t}{e_t} = \varepsilon_t - \pi_t \ .$$

- Therefore (19) can be differentiated as:

$$\dot{w}_t = -\left(\frac{1}{1-\bar{c}}\right)(\pi_t - \bar{\varepsilon}) + \left(\frac{1}{1-\bar{c}}\right)(\varepsilon_t - \bar{\varepsilon}) \ . \tag{20}$$

4 Summary

Equations (15), (17), (18) and (20) are summarized in equation (47) in the paper, which we reproduce here:

$$\begin{bmatrix} \dot{\psi}_t \\ \dot{\upsilon}_t \\ \dot{\pi}_t \\ \dot{\xi}_t \end{bmatrix} = \begin{bmatrix} -\delta & \delta & 0 & 0 \\ \frac{(\delta+r)^2}{\delta} & 0 & -\frac{(\delta+r)^2}{\delta} & (\delta+r)^2 \\ -(3\delta+2r) & 2\delta & (\delta+2r) & -2\delta(\delta+r) \\ 0 & 0 & -\frac{1}{1-\bar{c}} & 0 \end{bmatrix} \begin{bmatrix} (\psi_t-\bar{\varepsilon}) \\ (\upsilon_t-\bar{\varepsilon}) \\ (\pi_t-\bar{\varepsilon}) \\ (w_t-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{1-\bar{c}} \end{bmatrix} (\varepsilon_t - \bar{\varepsilon}) \;. \tag{21}$$