Online Appendix: Assessing DSGE Model Nonlinearities

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A QAR(1,1) Model

This section shows how to derive important moments for the QAR(1,1) model given by

$$y_t = \phi_1 y_{t-1} + \phi_2 s_{t-1}^2 + (1+\gamma s_{t-1})\sigma u_t, \quad u_t \sim iidN(0,1) \tag{A.1}$$

$$s_t = \phi_1 s_{t-1} + \sigma u_t, \quad |\phi_1| < 1 \tag{A.2}$$

by exploiting the recursively linear structure of the model. The model corresponds to (9) in the main text. To simplify the presentation, we dropped the tildes for ϕ_2, γ, and s.

A.1 Moments

We now derive the time-invariant mean and autocovariances for y_t, assuming the process is stationary and was initialized in the infinite past. Due to the recursively linear structure of the model, we begin with the derivation of the moments of s_t.

Moments of s_t. The process s_t in (A.2) is linear and has a moving average representation of the form

$$s_t = \sigma \sum_{j=0}^{\infty} \phi_1^j u_{t-j}.$$

The mean and the autocovariances of s_t are given by

$$\mathbb{E}[s_t] = 0, \quad \mu_{s^2} = \mathbb{E}[s_t^2] = \frac{\sigma^2}{1-\phi_1^2}, \quad \mathbb{E}[s_t s_{t-h}] = \phi_1^h \mu_{s^2}.$$

Since the innovations u_t are *iid* standard normal variates, we obtain the following third and fourth moments:

$$\mathbb{E}[s_t^3] = \sum_{j=0}^{\infty} \phi_1^{3j} \mathbb{E}[u_{t-j}^3] = 0, \quad \mathbb{E}[s_t^4] = \sum_{j=0}^{\infty} \phi_1^{4j} \mathbb{E}[u_{t-j}^4] = \frac{3\sigma^4}{1-\phi_1^4}.$$

Mean of y_t. Taking expectations on both sides of (A.1) we obtain

$$\mathbb{E}[y_t] = \phi_1 \mathbb{E}[y_{t-1}] + \phi_2 \mu_{s^2} + (1 + \gamma \mathbb{E}[s_{t-1}])\sigma \mathbb{E}[u_t] = \phi_1 \mathbb{E}[y_t] + \frac{\phi_2 \sigma^2}{1-\phi_1^2}.$$

Here we used the expression for μ_{s^2} obtained previously as well as the fact that u_t and s_{t-1} are independent. In turn,

$$\mu_y = \mathbb{E}[y_t] = \frac{\phi_2 \sigma^2}{(1-\phi_1)(1-\phi_1^2)}. \tag{A.3}$$

Variance of y_t. Consider the centered second moment of y_t:

$$\begin{aligned}
\mathbb{V}[y_t] &= \mathbb{E}\left[(\phi_1(y_{t-1}-\mu_y) + \phi_2(s_{t-1}^2 - \mu_{s^2}) + \sigma(1+\gamma s_{t-1})u_t)^2\right] \\
&= \mathbb{E}\big[\phi_1^2(y_{t-1}-\mu_y)^2 + \phi_2^2(s_{t-1}^2-\mu_{s^2})^2 + \sigma^2(1+\gamma s_{t-1})^2 u_t^2 \\
&\quad 2\phi_1\phi_2(y_{t-1}-\mu_y)(s_{t-1}^2-\mu_{s^2}) + 2\phi_2\sigma(s_{t-1}^2-\mu_{s^2})(1+\gamma s_{t-1})u_t \\
&\quad +2\phi_1\sigma(1+\gamma s_{t-1})(y_{t-1}-\mu_y)u_t\big] \\
&= \phi_1^2\mathbb{E}[(y_{t-1}-\mu_y)^2] + \phi_2^2\mathbb{E}[(s_{t-1}^2-\mu_{s^2})^2] + \sigma^2(1+\gamma^2\mu_{s^2}) \\
&\quad +2\phi_1\phi_2\mathbb{E}[(y_{t-1}-\mu_y)(s_{t-1}^2-\mu_{s^2})].
\end{aligned}$$

The time-invariant solution is

$$\mathbb{V}[y_t] = \frac{1}{1-\phi_1^2}\left[\phi_2^2\mathbb{V}[s_t^2] + \sigma^2(1+\gamma^2\mathbb{E}[s_t^2]) + 2\phi_1\phi_2 Cov[y_t, s_t^2]\right],$$

where

$$\begin{aligned}
Cov[y_t, s_t^2] &= \mathbb{E}\big[(\phi_1(y_{t-1}-\mu_y) + \phi_2(s_{t-1}^2-\mu_{s^2}) + (1+\gamma s_{t-1})\sigma u_t) \\
&\quad \times(\phi_1^2(s_{t-1}^2-\mu_{s^2}) + 2\phi_1\sigma s_{t-1}u_t + \sigma^2(u_t^2-1))\big] \\
&= \phi_1^3\mathbb{E}[(y_{t-1}-\mu_y)(s_{t-1}^2-\mu_{s^2})] + \phi_1^2\phi_2\mathbb{E}[(s_{t-1}^2-\mu_{s^2})^2] \\
&\quad +2\phi_1\gamma\sigma^2\mu_{s^2},
\end{aligned}$$

which implies

$$Cov[y_t, s_t^2] = \frac{1}{1-\phi_1^3}\left[\phi_1^2\phi_2\mathbb{V}[s_t^2] + 2\phi_1\gamma\sigma^2\mathbb{E}[s_t^2]\right].$$

Interestingly,

$$\begin{aligned} Cov[y_t, s_t] &= \mathbb{E}\big[(\phi_1(y_{t-1}-\mu_y) + \phi_2(s_{t-1}^2-\mu_{s^2}) + (1+\gamma s_{t-1})\sigma u_t)(\phi_1 s_{t-1} + \sigma u_t)\big] \\ &= \phi_1^2 Cov[y_{t-1}, s_{t-1}] + \sigma^2. \end{aligned}$$

All other terms drop out because $\mathbb{E}[u_t] = \mathbb{E}[s_t] = \mathbb{E}[s_t^3] = 0$. Thus, solving for the time-invariant solution leads to the "first order" variance expression

$$Cov[y_t, s_t] = \mathbb{E}[s_t^2] = \frac{\sigma^2}{1-\phi_1^2}.$$

Autocovariances of y_t. Consider $\mathbb{E}[(y_t-\mu_y)(y_{t-1}-\mu_y)]$:

$$\begin{aligned} Cov[y_t, y_{t-1}] &= \mathbb{E}\big[(\phi_1(y_{t-1}-\mu_y) + \phi_2(s_{t-1}^2-\mu_{s^2}) + (1+\gamma s_{t-1})\sigma u_t)(y_{t-1}-\mu_y)\big] \\ &= \phi_1\mathbb{V}[y_{t-1}] + \phi_2 Cov[y_{t-1}, s_{t-1}^2]. \end{aligned}$$

In general, higher-order autocovariances can be computed recursively:

$$\begin{aligned} Cov[y_t, y_{t-h}] &= \mathbb{E}\big[(\phi_1(y_{t-1}-\mu_y) + \phi_2(s_{t-1}^2-\mu_{s^2}) + (1+\gamma s_{t-1})\sigma u_t)(y_{t-h}-\mu_y)\big] \\ &= \phi_1 Cov[y_{t-1}, y_{t-h}] + \phi_2 Cov[y_{t-h}, s_{t-1}^2]. \end{aligned}$$

The term $Cov[y_{t-h}, s_{t-1}^2]$ can also be calculated recursively:

$$\begin{aligned} Cov[y_{t-h}, s_{t-1}^2] &= \mathbb{E}\big[(y_{t-h}-\mu_y)(\phi_1^2(s_{t-2}-\mathbb{E}[s_{t-2}^2]) + 2\phi_1 s_{t-2}\sigma u_{t-1} + \sigma(u_{t-1})^2 - 1)\big] \\ &= \phi_1^2 Cov[y_{t-h}, s_{t-2}^2]. \end{aligned}$$

A.2 Initialization and Identification

In order to compute the likelihood function recursively, it is necessary to initialize s_0. We write the joint distribution of observables, initial state, and parameters as

$$p(Y_{0:T}, \theta, s_0) = p(Y_{1:T}|y_0, s_0, \theta)p(y_0, s_0|\theta)p(\theta)$$

and use MCMC methods to generate draws from the posterior

$$p(\theta, s_0|Y_{0:T}) \propto p(Y_{1:T}|y_0, s_0, \theta)p(y_0, s_0|\theta)p(\theta).$$

We will approximate the distribution of (y_0, s_0) using a normal distribution

$$\left[\begin{array}{c} y_0 \\ s_0 \end{array}\right]\Bigg|\theta \sim N\left(\left[\begin{array}{c} \mu_y \\ \mu_s \end{array}\right], \left[\begin{array}{cc} \Sigma_{yy} & \Sigma_{ys} \\ \Sigma_{sy} & \Sigma_{ss} \end{array}\right]\right). \tag{A.4}$$

The moments of this normal distribution are calculated as follows. We will assume that the system was in its steady state in period $t = -T_*$, i.e., $s_{-T_*} = 0$ and $y_{-T_*} = \phi_0$. In principle, T_* could be infinite, but this will create some problems if $\phi_1 = 1$. In order to simplify the time subscripts a bit, we shift the time index by T_* periods. Starting from $s_0 = 0$ and $y_0 = \phi_0$, we will calculate the first and second moments of y_t, s_t, and s_t^2 recursively, starting with

$$\begin{array}{c} \mathbb{E}[s_0] = 0,\ \mathbb{E}[y_0] = \phi_0,\ \mathbb{V}[s_0] = 0,\ \mathbb{V}[y_0] = 0, \\ \mathrm{Cov}[y_0, s_0] = 0,\ \mathrm{Cov}[y_0, s_0^2],\ \mathbb{V}[s_0^2] = 0. \end{array} \tag{A.5}$$

The process for s_t is linear autoregressive of order one and we obtain

$$\mathbb{E}[s_t] = \phi_1 \mathbb{E}[s_{t-1}], \quad \mathbb{V}[s_t] = \phi_1^2 \mathbb{V}[s_{t-1}] + \sigma^2. \tag{A.6}$$

Since the innovations ϵ_t are *iid* standard normal variates, we see that the third moment is zero:

$$\mathbb{E}[s_t^3] = \sum_{j=0}^{t-1} \phi_1^{3j} \mathbb{E}[\epsilon_{t-j}^3] = 0.$$

Now consider

$$\begin{aligned} \mathbb{V}[s_t^2] &= \mathbb{E}[(s_t^2 - \mathbb{V}[s_t])^2] \\ &= \mathbb{E}[(\phi_1^2(s_{t-1}^2 - \mathbb{V}[s_{t-1}]) + 2\phi_1 s_{t-1}\sigma\epsilon_t + \sigma^2(\epsilon_t^2 - 1))^2] \\ &= \phi_1^4 \mathbb{V}[s_{t-1}^2] + 4\phi_1^2\sigma^2 \mathbb{V}[s_{t-1}] + 2\sigma^4. \end{aligned} \tag{A.7}$$

A formula for the mean of y_t is obtained by taking expectations of the observation equation:

$$\mathbb{E}[y_t] = \phi_0(1 - \phi_1) + \phi_1 \mathbb{E}[y_{t-1}] + \phi_2 \mathbb{V}[s_{t-1}]. \tag{A.8}$$

The covariance between y_t and s_t is given by

$$\begin{aligned} \mathrm{Cov}[y_t, s_t] &= \mathbb{E}[(y_t - \mathbb{E}[y_t])s_t] \qquad (A.9) \\ &= \mathbb{E}\big[\big(\phi_1(y_{t-1} - \mathbb{E}[y_{t-1}]) + \phi_2(s_{t-1}^2 - \mathbb{E}[s_{t-1}^2]) + (1+\gamma s_{t-1})\sigma\epsilon_t\big)\big(\phi_1 s_{t-1} + \sigma\epsilon_t\big)\big] \\ &= \phi_1^2 \mathrm{Cov}[y_{t-1}, s_{t-1}] + \sigma^2. \end{aligned}$$

All other terms drop out because the first and third moments of s_{t-1} and ϵ_t are equal to zero. The covariance between y_t and s_t^2 is given by

$$\begin{aligned} \mathrm{Cov}[y_t, s_t^2] &= \mathbb{E}[(y_t - \mathbb{E}[y_t])(s_t^2 - \mathbb{V}[s_t])] \qquad (A.10) \\ &= \mathbb{E}\big[(\phi_1(y_{t-1} - \mathbb{E}[y_{t-1}]) + \phi_2(s_{t-1}^2 - \mathbb{V}[s_{t-1}]) + (1+\gamma s_{t-1})\sigma\epsilon_t) \\ &\quad \times(\phi_1^2(s_{t-1}^2 - \mathbb{V}[s_{t-1}]) + 2\phi_1\sigma s_{t-1}\epsilon_t + \sigma^2(\epsilon_t^2 - 1))\big] \\ &= \phi_1^3 \mathrm{Cov}[y_{t-1}, s_{t-1}^2] + \phi_1^2\phi_2\mathbb{V}[s_{t-1}^2] + 2\phi_1\gamma\sigma^2\mathbb{E}[s_{t-1}^2]. \end{aligned}$$

The variance of y_t can be computed as follows:

$$\begin{aligned} \mathbb{V}[y_t] &= \mathbb{E}\left[(\phi_1(y_{t-1} - \mathbb{E}[y_{t-1}] + \phi_2(s_{t-1}^2 - \mathbb{V}[s_{t-1}]) + \sigma(1+\gamma s_{t-1})\epsilon_t)^2\right] \qquad (A.11) \\ &= \phi_1^2\mathbb{V}[y_{t-1}] + \phi_2^2\mathbb{V}[s_{t-1}^2] + \sigma^2(1+\gamma^2\mathbb{V}[s_{t-1}]) \\ &\quad +2\phi_1\phi_2\mathrm{Cov}[y_{t-1}, s_{t-1}^2]. \end{aligned}$$

We can iterate Equations (A.6) to (A.11) forward for T_* periods to obtain the moments for the initial distribution of (y_0, s_0) in (A.4).

Note, that for $\gamma = \phi_2 = 0$, s_0 and y_0 become perfectly correlated conditional on θ since for a linear model $y_0 = s_0 + \phi_0$. This may affect our posterior sampler when we include s_0 into the parameter vector. To avoid the singularity we add a small constant to the covariance matrix of (y_0, s_0).

A.3 MCMC Implementation

The RWM algorithm mentioned in Section 3.3 is used to implement the posterior inference. Using a preliminary covariance for the proposal distribution in the RWM algorithm that is constructed from the prior variance of the QAR parameters, we generate an initial 100,000 draws from the posterior. Based on the last 50,000 draws, we compute a covariance matrix that replaces the preliminary covariance matrix of the proposal distribution. We then continue the chain, generating an additional 60,000 draws and retaining the last 50,000 to construct summary statistics for the posterior.

A.4 Detailed Estimation Results

Table A-1: Prior Distribution for QAR(1,1) Model, Samples Starting in 1960

	GDP Growth	Wage Growth	Inflation	Fed Funds Rate
ϕ_0	$N\big(0.48, 2\big)$	$N\big(1.18, 2\big)$	$N\big(2.38, 2\big)$	$N\big(2.50, 2\big)$
ϕ_1	$N^{\dagger}(0.36, 0.5)$	$N^{\dagger}(-0.02, 0.5)$	$N^{\dagger}(0.00, 0.5)$	$N^{\dagger}(0.66, 0.5)$
σ	$IG(1.42, 4)$	$IG(0.82, 4)$	$IG(1.87, 4)$	$IG(0.58, 4)$
ϕ_2	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$
γ	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$

Notes: (†) The prior for ϕ_1 is truncated to ensure stationarity. The IG distribution is parameterized such that $p_{IG}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$.

Table A-2: Prior Distribution for QAR(1,1) Model, Samples Starting in 1984

	GDP Growth	Wage Growth	Inflation	Fed Funds Rate
ϕ_0	$N\big(0.43, 2\big)$	$N\big(1.58, 2\big)$	$N\big(4.38, 2\big)$	$N\big(6.08, 2\big)$
ϕ_1	$N^{\dagger}(0.28, 0.5)$	$N^{\dagger}(0.34, 0.5)$	$N^{\dagger}(0.85, 0.5)$	$N^{\dagger}(0.94, 0.5)$
σ	$IG(1.33, 4)$	$IG(0.88, 4)$	$IG(1.83, 4)$	$IG(1.45, 4)$
ϕ_2	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$
γ	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$

Notes: (†) The prior for ϕ_1 is truncated to ensure stationarity. The IG distribution is parameterized such that $p_{IG}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$.

Table A-3: Posterior Estimates for QAR(1,1) Model, 1960:Q1 to 1983:Q4

Data	ϕ_0	ϕ_1	ϕ_2	γ	σ	s_0
GDP	0.42	0.28	-0.02	-0.05	1.16	1.42
	[0.11 , 0.69]	[0.11 , 0.46]	[-0.14 , 0.09]	[-0.17 , 0.06]	[0.91 , 1.53]	[1.02 , 1.85]
WAGE	1.75	0.41	-0.05	0.04	0.52	0.89
	[1.49 , 1.98]	[0.23 , 0.58]	[-0.13 , 0.04]	[-0.05 , 0.15]	[0.40 , 0.68]	[0.63 , 1.15]
INFL	4.24	0.87	-0.01	0.16	1.52	-1.97
	[2.28 , 5.84]	[0.80 , 0.95]	[-0.08 , 0.07]	[0.04 , 0.27]	[1.08 , 2.12]	[-4.68 , 0.79]
FFR	4.84	0.92	0.02	0.38	0.62	-1.56
	[0.86 , 6.75]	[0.88 , 0.96]	[-0.05 , 0.05]	[0.30 , 0.47]	[0.41 , 1.00]	[-4.21 , 0.14]

Notes: We report posterior means and 90% equal-tail-probability credible sets in brackets.

Table A-4: Posterior Estimates for QAR(1,1) Model, 1960:Q1 to 2007:Q4

Data	ϕ_0	ϕ_1	ϕ_2	γ	σ	s_0
GDP	0.48	0.29	-0.02	-0.06	0.69	1.37
	[0.33 , 0.63]	[0.16 , 0.41]	[-0.07 , 0.04]	[-0.13 , 0.01]	[0.58 , 0.82]	[1.19 , 1.56]
WAGE	1.41	0.44	-0.03	0.12	0.48	1.22
	[1.25 , 1.59]	[0.33 , 0.55]	[-0.09 , 0.02]	[0.05 , 0.20]	[0.40 , 0.57]	[1.00 , 1.42]
INFL	3.51	0.85	-0.01	0.23	1.06	-1.31
	[2.74 , 4.47]	[0.79 , 0.91]	[-0.06 , 0.05]	[0.16 , 0.31]	[0.81 , 1.38]	[-2.90 , 0.31]
FFR	2.96	0.96	0.04	0.44	0.28	-0.74
	[2.16 , 4.16]	[0.95 , 0.97]	[0.02 , 0.06]	[0.37 , 0.52]	[0.22 , 0.42]	[-1.27 , 0.45]

Notes: We report posterior means and 90% equal-tail-probability credible sets in brackets.

Table A-5: Posterior Estimates for QAR(1,1) Model, 1960:Q1 to 2012:Q4

Data	ϕ_0	ϕ_1	ϕ_2	γ	σ	s_0
GDP	0.45	0.33	-0.03	-0.07	0.68	1.41
	[0.28 , 0.60]	[0.22 , 0.44]	[-0.08 , 0.03]	[-0.14 , 0.00]	[0.58 , 0.81]	[1.19 , 1.61]
WAGE	1.29	0.43	-0.01	0.08	0.54	1.31
	[1.12 , 1.46]	[0.32 , 0.53]	[-0.06 , 0.04]	[0.01 , 0.15]	[0.46 , 0.63]	[1.11 , 1.50]
INFL	3.23	0.84	0.02	0.22	1.09	-1.26
	[2.55 , 4.16]	[0.78 , 0.90]	[-0.04 , 0.09]	[0.15 , 0.30]	[0.87 , 1.36]	[-2.82 , 0.22]
FFR	3.54	0.96	-0.01	0.41	0.22	0.43
	[2.29 , 5.06]	[0.94 , 0.97]	[-0.02 , 0.00]	[0.33 , 0.50]	[0.13 , 0.37]	[-0.94 , 1.47]

Notes: We report posterior means and 90% equal-tail-probability credible sets in brackets.

Table A-6: Posterior Estimates for QAR(1,1) Model, 1984:Q1 to 2007:Q4

Data	ϕ_0	ϕ_1	ϕ_2	γ	σ	s_0
GDP	0.57	0.26	-0.07	0.01	0.25	1.06
	[0.44, 0.70]	[0.10 , 0.44]	[-0.13 , -0.02]	[-0.10 , 0.11]	[0.20 , 0.32]	[0.91,1.21]
WAGE	1.09	0.24	-0.06	0.07	0.41	0.10
	[0.93,1.21]	[0.06,0.42]	[-0.12,0.02]	[-0.03,0.17]	[0.32,0.53]	[-0.09,0.29]
INFL	2.72	0.63	-0.06	0.07	0.68	2.42
	[2.30,3.13]	[0.48,0.78]	[-0.14,0.04]	[-0.06,0.19]	[0.52,0.89]	[1.76,2.93]
FFR	9.80	0.91	-0.16	0.08	0.22	0.79
	[8.68,11.56]	[0.87,0.93]	[-.23,-.10]	[-0.03,0.17]	[0.15,0.32]	[-0.26,1.64]

Notes: We report posterior means and 90% equal-tail-probability credible sets in brackets.

Table A-7: Posterior Estimates for QAR(1,1) Model, 1984:Q1 to 2012:Q4

Data	ϕ_0	ϕ_1	ϕ_2	γ	σ	s_0
GDP	0.53	0.36	-0.09	-0.07	0.28	1.09
	[0.38 , 0.66]	[0.22 , 0.52]	[-0.15 , -0.03]	[-0.17 , -0.00]	[0.23 , 0.35]	[0.87 , 1.28]
WAGE	0.98	0.18	-0.04	0.03	0.48	0.20
	[0.83 , 1.14]	[0.02 , 0.36]	[-0.10 , 0.04]	[-0.06 , 0.12]	[0.38 , 0.60]	[0.03 , 0.37]
INFL	2.51	0.63	-0.02	0.07	0.76	2.54
	[2.12 , 2.93]	[0.48 , 0.77]	[-0.10 , 0.06]	[-0.03 , 0.19]	[0.61 , 0.97]	[1.80 , 3.00]
FFR	10.00	0.92	-0.17	0.01	0.19	1.00
	[8.72 , 11.43]	[0.90 , 0.94]	[-0.25 , -0.12]	[-0.05 , 0.11]	[0.15 , 0.29]	[0.05 , 1.40]

Notes: We report posterior means and 90% equal-tail-probability credible sets in brackets.

B The DSGE Model

B.1 First-Order Conditions

Intermediate Goods Producers. Taking as given nominal wages, final good prices, the demand schedule for intermediate products, and technological constraints, firm j chooses its labor inputs $H_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits. After using the production function to substitute our $Y_t(j)$ from the present value of future profits in (24) (see main text) we can write the objective function of the firm as

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}}\left(1-\Phi_p\left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)}\right)\right)A_{t+s}H_{t+s}(j)-\frac{1}{P_{t+s}}W_{t+s}H_{t+s}(j)\right)\right]. \tag{A.12}$$

This objective function is maximized with respect to $H_t(j)$ and $P_t(j)$ subject to

$$A_{t+s}H_{t+s}(j)=\left(\frac{P_t(j)}{P_t}\right)^{-1/\lambda_{p,t}}Y_{t+s}.$$

We use $\mu_{t+s}\beta^s Q_{t+s|t}$ to denote the Lagrange multiplier associated with this constraint. Setting $Q_{t|t}=1$, the first-order condition with respect to $P_t(j)$ is given by

$$\begin{aligned} 0 \;=\; & \frac{1}{P_t}\left(1-\Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right)\right)A_tH_t(j)-\frac{P_t(j)}{P_tP_{t-1}(j)}\Phi_p'\left(\frac{P_t(j)}{P_{t-1}(j)}\right)A_tH_t(j) \qquad \text{(A.13)}\\ & -\frac{\mu_t}{\lambda_{p,t}P_t}\left(\frac{P_t(j)}{P_t}\right)^{-1/\lambda_{p,t}-1}Y_t+\beta\mathbb{E}_t\left[Q_{t+1|t}\frac{P_{t+1}^2(j)}{P_{t+1}P_t^2(j)}\Phi_p'\left(\frac{P_{t+1}(j)}{P_t(j)}\right)A_{t+1}H_{t+1}(j)\right]. \end{aligned}$$

Taking first-order conditions with respect to $H_t(j)$ yields

$$\frac{W_t}{P_t}=\frac{P_t(j)}{P_t}\left(1-\Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right)\right)A_t-\mu_tA_t. \tag{A.14}$$

Households. The first-order condition with respect to consumption is given by

$$P_t\lambda_t=\left(\frac{C_t(k)}{A_t}\right)^{-\tau}\frac{1}{A_t}. \tag{A.15}$$

We define

$$Q_{t+1|t}=\frac{\lambda_{t+1}P_{t+1}}{\lambda_tP_t}. \tag{A.16}$$

Using this definition, the first-order condition for bond holdings becomes

$$1 = \beta \mathbb{E}_t \left[Q_{t+1|t} \frac{R_t}{\pi_{t+1}} \right]. \tag{A.17}$$

Member k is a monopolistic competitor with respect to his wage choice. Taking into account the demand for labor of type k the relevant portion of the utility function for the wage decision is

$$I\!E_t \left[\sum_{s=0}^{\infty} \beta^s \left(\cdots - \chi_H \frac{1}{1+1/\nu} \left(\frac{W_{t+s}(k)}{W_{t+s}} \right)^{-(1+1/\nu)/\lambda_w} H_t^{1+1/\nu} \right) \right].$$

The relevant portion of the budget constraint after substituting $H_{t+s}(k)$ by the labor demand schedule is

$$\cdots = W_{t+s}(k) \left(\frac{W_{t+s}(k)}{W_{t+s}} \right)^{-1/\lambda_w} H_{t+s} \left(1 - \Phi_w \left(\frac{W_{t+s}(k)}{W_{t+s-1}(k)} \right) \right) + \cdots ,$$

where the demand for aggregated labor services H_{t+s} is taken as given. Taking first-order conditions with respect to $W_t(k)$ yields

$$0 = \frac{\chi_H}{\lambda_w W_t} \left(\frac{W_t(k)}{W_t} \right)^{-\frac{1+1/\nu}{\lambda_w} - 1} H_t^{1+1/\nu} + \lambda_t \left(\frac{W_t(k)}{W_t} \right)^{-1/\lambda_w} H_t \left(1 - \Phi_w \left(\frac{W_t(k)}{W_{t-1}(k)} \right) \right) \tag{A.18}$$

$$- \frac{\lambda_t}{\lambda_w} \frac{W_t(k)}{W_t} \left(\frac{W_t(k)}{W_t} \right)^{-1/\lambda_w - 1} H_t \left(1 - \Phi_w \left(\frac{W_t(k)}{W_{t-1}(k)} \right) \right) \tag{A.19}$$

$$- \lambda_t \frac{W_t(k)}{W_{t-1}(k)} \left(\frac{W_t(k)}{W_t} \right)^{-1/\lambda_w} H_t \Phi_w' \left(\frac{W_t(k)}{W_{t-1}(k)} \right)$$

$$+ \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{W_{t+1}^2(k)}{W_t^2(k)} \left(\frac{W_{t+1}(k)}{W_{t+1}} \right)^{-1/\lambda_w} H_{t+1} \Phi_w' \left(\frac{W_{t+1}(k)}{W_t(k)} \right) \right].$$

B.2 Equilibrium Relationships

We consider the symmetric equilibrium in which all intermediate-goods-producing firms, as well as households, make identical choices when solving their optimization problem. Therefore, we can drop the index k and j. In slight abuse of notation, let $\Delta X_t = X_t / X_{t-1}$ and $\pi_t = \Delta P_t$. We use $w_t = W_t / P_t$ to denote the real wage. Since the non-stationary technology process A_t induces a stochastic trend in output, consumption, and real wages, it is convenient to express the model in terms of detrended variables $y_t = Y_t / A_t$, $c_t = C_t / A_t$ and $\tilde{w}_t = w_t / A_t$.

Intermediate Goods Producers. Using the above notation, multiplying (A.13) by P_t, and replacing Y_t by $A_t y_t$. we can simplify the first-order condition for $P_t(j)$ as follows:

$$0 = \big(1 - \Phi_p(\pi_t)\big) A_t y_t - \pi_t \Phi_p'(\pi_t) A_t y_t - \frac{\mu_t}{\lambda_{p,t}} A_t y_t + \beta \mathbb{E}_t \big[Q_{t+1|t} \pi_{t+1} \Phi_p'(\pi_{t+1}) A_{t+1} y_{t+1} \big].$$

Dividing by $A_t y_t$ and replacing A_{t+1}/A_t by $\gamma \exp(z_{t+1})$ we obtain

$$0 = \big(1 - \Phi_p(\pi_t)\big) - \pi_t \Phi_p'(\pi_t) - \frac{\mu_t}{\lambda_{p,t}} + \beta \mathbb{E}_t \big[Q_{t+1|t} \pi_{t+1} \Phi_p'(\pi_{t+1}) \Delta y_{t+1} \gamma \exp(z_{t+1}) \big].$$

We proceed by rewriting (A.14) as

$$\tilde{w}_t = \big(1 - \Phi_p(\pi_t)\big) - \mu_t. \tag{A.20}$$

Households. In terms of detrended consumption we can express $Q_{t+1|t}$ as

$$Q_{t+1|t} = \left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma} \exp(-z_{t+1}). \tag{A.21}$$

The consumption Euler equation remains unchanged:

$$1 = \beta \mathbb{E}_t \left[Q_{t+1|t} \frac{R_t}{\pi_{t+1}} \right]. \tag{A.22}$$

We now divide (A.19) by λ_t and replace λ_t by $c_t^{-\tau}/(A_t P_t)$:

$$\begin{aligned} 0 \;\; = \;\; & \frac{\chi_H}{\lambda_w} \frac{1}{\tilde{w}_t} c_t^{\tau} H_t^{1+1/\nu} + H_t \big(1 - \Phi_w(\pi_t \Delta w_t)\big) - \frac{1}{\lambda_w} H_t \big(1 - \Phi_w(\pi_t \Delta w_t)\big) \\ & - \pi_t \Delta w_t H_t \Phi_w'(\pi_t \Delta w_t) + \beta \mathbb{E}_t \left[Q_{t+1|t} \pi_{t+1} \Delta w_{t+1}^2 H_{t+1} \Phi_w'(\pi_{t+1} \Delta w_{t+1}) \right]. \end{aligned}$$

Aggregate Resource Constraint. The aggregate production function (in terms of detrended output) is

$$y_t = H_t. \tag{A.23}$$

The intermediate goods producers' dividend payments to the households are given by

$$D_t = \big(1 - \Phi_p(\pi_t)\big) Y_t - w_t H_t. \tag{A.24}$$

Combining the household budget constraint and the government budget constraint and detrending all variables leads to the aggregate resource constraint

$$c_t + \zeta y_t = \big(1 - \Phi_p(\pi_t)\big) y_t - \tilde{w}_t y_t \Phi_w(\pi_t \Delta w_t),$$

where $\Delta w_t = \Delta \tilde{w}_t \gamma \exp(z_t)$.

The model economy has a unique steady state in terms of the detrended variables that is attained if the innovations $\epsilon_{R,t}$, $\epsilon_{g,t}$, and $\epsilon_{z,t}$ are zero at all times. The steady-state inflation π equals the target rate π^* and

$$R = \frac{\gamma}{\beta}\pi^*,\ \mu = \lambda_p,\ c = \left(\frac{(1-\lambda_p)(1-\lambda_w) g^{-\frac{1}{\nu}}}{\chi_H} \right)^{\frac{1}{\tau + 1/\nu}},\ y = g\tilde{c},\ H = y,\ \tilde{w} = (1-\lambda_p).$$

B.3 Posterior Simulator

We first estimate a log-linearized version of the DSGE model using the random walk Metropolis (RWM) algorithm described in An and Schorfheide (2007). Using the same covariance matrix for the proposal distribution as for the linearized DSGE model, we then run the RWM algorithm based on the likelihood function associated with the second-order approximation of the DSGE model. The covariance matrix of the proposal distribution is scaled such that the RWM algorithm has an acceptance rate of approximately 50%. We use 80,000 particles to approximate the likelihood function of the nonlinear DSGE model, while the variance of measurement errors is set to 10% of the sample variance of the observables. We generate 120,000 draws from the posterior distribution of the nonlinear DSGE model. The summary statistics reported in Table 3 in the main paper are based on the last 100,000 draws of this sequence.

Table A-8: Posterior Estimates for DSGE Model Parameters: Linear Model

	1960:Q1 to 2007:Q4		1984:Q1 to 2007:Q4	
Parameter	Mean	90% Interval	Mean	90% Interval
$400\left(\frac{1}{\beta}-1\right)$	0.48	[0.06, 1.01]	1.31	[0.60, 2.17]
π^A	3.46	[2.94, 3.97]	2.80	[2.33, 3.29]
γ^A	1.86	[1.39, 2.34]	1.88	[1.53, 2.24]
τ	6.54	[4.37, 9.24]	4.78	[2.57, 8.70]
ν	0.09	[0.06, 0.13]	0.08	[0.03, 0.15]
$\kappa(\varphi_p)$	0.01	[0.01, 0.02]	0.18	[0.09, 0.30]
φ_w	62.33	[44.48, 83.14]	14.89	[6.15, 25.88]
ψ_w		N/A		
ψ_p		N/A		
ψ_1	1.45	[1.24, 1.68]	2.67	[2.10, 3.30]
ψ_2	0.80	[0.54, 1.09]	0.76	[0.41, 1.11]
ρ_r	0.77	[0.73, 0.82]	0.71	[0.61, 0.79]
ρ_g	0.97	[0.96, 0.98]	0.96	[0.93, 0.98]
ρ_z	0.26	[0.10, 0.41]	0.07	[0.01, 0.19]
ρ_p	0.99	[0.98, 0.99]	0.93	[0.87, 0.98]
$100\sigma_r$	0.18	[0.14, 0.22]	0.18	[0.13, 0.25]
$100\sigma_g$	0.65	[0.44, 0.95]	0.76	[0.39, 1.34]
$100\sigma_z$	0.75	[0.64, 0.85]	0.47	[0.37, 0.56]
$100\sigma_p$	15.28	[12.66, 18.18]	7.63	[5.96, 9.48]

Notes: Estimation sample is 1984:Q1 to 2010:Q4. For 90% credible interval we are reporting the 5th and 95th percentile of the posterior distribution.