Online Appendix for "Buy-it-now or Take-a-chance: Price Discrimination through Randomized Auctions"

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In this online appendix, we derive the optimal mechanism for the environment in our paper, using the techniques developed by Myerson (1981). We also show that there is a unique trembling hand perfect equilibrium for our environment under complete information.

Optimal Mechanisms

We start with the optimal mechanism for the two-type case.

Proposition 1 *For the two-type case, if $v_L \geq \alpha v_H$, the BIN-TAC mechanism with $d = n$ and $p = v_H - \frac{1}{n}(v_H - v_L)$ is optimal. If $v_L < \alpha v_H$, then the optimal mechanism is the BIN-TAC mechanism with $d = \infty$ and $p = v_H$.*

Proof Consider an (optimal) incentive compatible mechanism for the two-type case. Let $q_{i,L}$ and $q_{i,H}$ be the probability that the mechanism allocates the item to bidder i if her type is respectively low and high. We assume, without loss of generality, that if the mechanism allocates the item to a low-type bidder, it charges the bidder a price of v_L. Let $p_{i,H}$ be the *expected payment* of bidder i to the mechanism if her type is high. By the incentive compatibility constraint we have $q_{i,H}v_H - p_{i,H} \geq q_{i,L}(v_H - v_L)$. Hence, for an optimal mechanism we get

$$p_{i,H} = q_{i,H}v_H - q_{i,L}(v_H - v_L)$$

Therefore, the expected revenue of the mechanism from buyer i is equal to:

$$\alpha(q_{i,H}v_H - q_{i,L}(v_H - v_L)) + (1-\alpha)q_{i,L}v_L \quad = \quad \alpha v_H q_{i,H} + q_{i,L}(v_L - \alpha v_H)$$

Observe that for an optimal mechanism $q_{i,L}$ is equal to 0 if $v_L < \alpha v_H$. In this case, $p_{i,H} = q_{i,H}v_H$ which is equivalent to a posted price mechanism with price v_H or a BIN-TAC mechanism with $d = \infty$. In the rest of the proof, we assume $v_L \geq \alpha v_H$.

The following LP gives an upper-bound on the revenue of any optimal solution.

$$\begin{array}{ll} \text{Maximize}_{\{q_{i,H},q_{i,L}\}} & \alpha v_H \sum_{i=1}^n q_{i,H} + (v_L - \alpha v_H)\sum_{i=1}^n q_{i,L} \\ \text{Subject to:} & \sum_{i=1}^n \big(\alpha q_{i,H} + (1-\alpha)q_{i,L}\big) \leq 1 \\ & \alpha \sum_{i=1}^n q_{i,H} \leq 1 - (1-\alpha)^n \end{array}$$

The first constraint holds because we can allocate the item to at most to one bidder. The second constraint follows from the $1-(1-\alpha)^n$ bound on the probability that there exists a bidder of high-type. Since $v_H > \frac{v_L - \alpha v_H}{1-\alpha}$, the optimal solution of the program above is given by:

$$q_{i,L} = \frac{1}{n}(1-\alpha)^{n-1} \quad \& \quad q_{i,H} = \frac{1}{\alpha n}\left(1-(1-\alpha)^n\right)$$

This corresponds to the mechanism that allocates the item at random to the bidder with the high type and if there's no high bids then it randomly allocates the item at price v_L. Therefore, the revenue of any optimal mechanism is bounded by:

$$v_H\left(1-(1-\alpha)^n\right) + (v_L - \alpha v_H)(1-\alpha)^{n-1}$$

On the other hand, the revenue of BIN-TAC with $d = n$ (and $p = v_H - \frac{1}{n}(v_H - v_L)$) is equal to:

$$\begin{aligned} & (1-\alpha)^{n-1}v_L + n\alpha(1-\alpha)^{n-1}p + \big(1-(1-\alpha)^n - n\alpha(1-\alpha)^{n-1}\big)\,v_H \\ = \; & (1-\alpha)^{n-1}v_L + n\alpha(1-\alpha)^{n-1}v_H - \alpha(1-\alpha)^{n-1}(v_H - v_L) + \big(1-(1-\alpha)^n - n\alpha(1-\alpha)^{n-1}\big)\,v_H \\ = \; & (1-\alpha)^{n-1}(v_L - \alpha v_H) + \left(1-(1-\alpha)^n\right)v_H \end{aligned}$$

Therefore, BIN-TAC with $d = n$ is optimal. $\square$

We now consider the general model. Since the payment structure is well-known given the ironed virtual valuations, the challenge is to compute the ironed virtual values. This approach requires the distribution of values, F, to be strictly increasing. Hence, we consider the following distribution of the values.

$$f_\varepsilon(x) = \begin{cases} \beta f_L(x) & x \in [\underline{\omega}_L, \overline{\omega}_L] \\ \varepsilon & x \in (\overline{\omega}_L, \underline{\omega}_H) \\ f_H(x)\alpha & x \in [\underline{\omega}_H, \overline{\omega}_H] \end{cases}$$

$$F_\varepsilon(x) = \begin{cases} \beta F_L(x) & x \in [\underline{\omega}_L, \overline{\omega}_L] \\ \beta + \varepsilon(x - \overline{\omega}_L) & x \in (\overline{\omega}_L, \underline{\omega}_H) \\ (1-\alpha) + \alpha F_H(x - \underline{\omega}_H) & \in [\underline{\omega}_H, \overline{\omega}_H] \end{cases}$$

where $\beta + \varepsilon(\underline{\omega}_H - \overline{\omega}_L) + \alpha = 1$. As ε tends to 0 we get the original model back. We need to "iron" the virtual values. For $q \in [0, 1]$, let $F_\varepsilon^{-1}(q)$ be the inverse of $F_\varepsilon(\cdot)$. Define:

$$h(q) = F_\varepsilon^{-1}(q) - \frac{1-q}{f_\varepsilon\left(F_\varepsilon^{-1}(q)\right)}$$

$$H(q) = \int_0^q h(y)dy$$

$$G(q) = \min_{\lambda, r_1, r_2 \in [0,1], \lambda r_1 + (1-\lambda) r_2 = q} \{\lambda H(r_1) + (1-\lambda) H(r_2)\}$$

This implies that $G(\cdot)$ is the highest convex function on $[0, 1]$ such that $G(q) \leq H(q)$ for every q. Define $\phi(v) = G'(F(v))$ as the *virtual value* of type v. The optimal mechanism randomly allocates the item to one of the bidders with the highest positive virtual value. We first show that the ironed virtual values are the same as the original virtual valuations, except for a set of quantiles between q^* and $(1 - \alpha)$:

Lemma 1 *Let $q^* = (1-\alpha)\widetilde{v}$ and $\widetilde{v}$ be the solution of*

$$-F^2(\widetilde{v}) + (2-\alpha)F(\widetilde{v}) + \alpha(\underline{\omega}_H - \widetilde{v})f(\widetilde{v}) = 1 - \alpha.$$

Then, by assumption $\psi(\overline{\omega}_L) \le \psi(\underline{\omega}_H)$, as $\varepsilon \to 0$,

$$G'(q) \;=\; \begin{cases} h(q) & q \in [0, q^*] \\ h(q^*) & q \in (q^*, 1-\alpha) \\ h(q) & q \in [1-\alpha, 1] \end{cases}$$

Proof First note that $H(q)$ is convex in $[0, \beta]$ because of the assumption that $x - \frac{1-F(x)}{f(x)}$ is increasing in $[\underline{\omega}_L, \overline{\omega}_L]$. It is also decreasing in $[0, q_0]$ and increasing in $[q_0, \beta]$, where $q_0 = F(r^*)$ is the minimum of $H(\cdot)$ in this range. Also, observe that $H(q)$ is decreasing in $[\beta, 1-\alpha]$ because $h(q) < 0$ in this interval. In addition, by the assumption that $\psi(v)$ is increasing over the regions $[\underline{\omega}_L, \overline{\omega}_L]$ and $[\underline{\omega}_H, \overline{\omega}_H]$, $H(q)$ is increasing and convex in $[1-\alpha, 1]$. Therefore, $G(\cdot)$ includes the tangent line from the point $(1-\alpha, H(1-\alpha))$ to $H(q)$ in $[0, \beta]$. Let q^* be the tangent point. We have

$$G(q) \;=\; \begin{cases} H(q) & q \in [0, q^*] \\ \frac{(q-q^*)H(q^*) + (1-\alpha-q)H(1-\alpha)}{1-\alpha-q^*} & q \in (q^*, 1-\alpha) \\ H(q) & q \in [1-\alpha, 1] \end{cases}$$

which immediately leads to the claim. □

In the rest we compute q^*. For $q \in [0, \beta]$,

$$\begin{aligned} H(q) &= \int_0^q \left(F_\varepsilon^{-1}(y) - \frac{1-y}{f_\varepsilon\left(F_\varepsilon^{-1}(y)\right)} \right) dy \\ &= \int_{\underline{\omega}_L}^{F^{-1}(q)} \left(x - \frac{1-F_\varepsilon(x)}{f_\varepsilon(x)} \right) f_\varepsilon(x) dx \\ &= \int_{\underline{\omega}_L}^{F^{-1}(q)} \left((x f_\varepsilon(x) + F_\varepsilon(x)) - 1 \right) dx \\ &= (q-1) F_\varepsilon^{-1}(q) + \underline{\omega}_L \end{aligned}$$

In particular,

$$H(\beta) = (\beta - 1)\overline{\omega}_L + \underline{\omega}_L$$

For $q \in (\beta, 1-\alpha)$, because $h(q) = \frac{2q-(1+\beta)}{\varepsilon} + \overline{\omega}_L$, we get

$$\begin{aligned}
H(q) &= H(\beta) + \left[\frac{x^2 - (1+\beta-\varepsilon\overline{\omega}_L)x}{\varepsilon}\right]_\beta^q \\
&= (\beta-1)\overline{\omega}_L + \underline{\omega}_L + \frac{q^2-\beta^2-(q-\beta)(1+\beta-\varepsilon\overline{\omega}_L)}{\varepsilon} \\
&= (q-1)\overline{\omega}_L + \underline{\omega}_L + (q-\beta)\frac{q-1}{\varepsilon} \qquad (1) \\
H(1-\alpha) &= -\alpha\overline{\omega}_L + \underline{\omega}_L + (1-\alpha-\beta)\frac{-\alpha}{\varepsilon} = -\alpha\underline{\omega}_H + \underline{\omega}_L \qquad (2)
\end{aligned}$$

To iron the distribution, we compute the tangent from $H(1-\alpha)$ to $H(q)$, for $q \in [0, 1-\alpha]$. Note that if q^* is the tangent point then

$$h(q^*) = \frac{H(1-\alpha) - H(q^*)}{1-\alpha-q^*} \qquad (3)$$

Observe that by Eq. (1) we have

$$\begin{aligned}
&\frac{H(1-\alpha) - H(q^*)}{1-\alpha-q^*} \\
=\ &\frac{(-\alpha\underline{\omega}_H + \underline{\omega}_L) - ((q^*-1)F_\varepsilon^{-1}(q^*) + \underline{\omega}_L)}{1-\alpha-q^*} \\
=\ &\frac{-\alpha\underline{\omega}_H - (q^*-1)F_\varepsilon^{-1}(q^*)}{1-\alpha-q^*}
\end{aligned}$$

Let $\widetilde{v} = F_\varepsilon^{-1}(q^*)$, i.e., $q^* = F_\varepsilon(\widetilde{v}) = \beta F_L(\widetilde{v})$. Therefore,

$$\frac{H(1-\alpha) - H(q^*)}{1-\alpha-q^*} = \frac{-\alpha\underline{\omega}_H - (F_\varepsilon(\widetilde{v})-1)\widetilde{v}}{1-\alpha-F_\varepsilon(\widetilde{v})}$$

$$h(q^*) = \widetilde{v} - \frac{1-F_\varepsilon(\widetilde{v})}{f_\varepsilon(\widetilde{v})}$$

As $\varepsilon \to 0$, the $F_\varepsilon(\cdot) \to F(\cdot)$. Plugging into Eq. (3) we get

$$\begin{aligned}
&(\widetilde{v}f(\widetilde{v}) - 1 + F(\widetilde{v}))(1-\alpha-F(\widetilde{v})) \\
=\ &f(\widetilde{v})\big(-\alpha\underline{\omega}_H - (F(\widetilde{v})-1)\widetilde{v}\big)
\end{aligned}$$

Hence, re-arranging the terms,

$$-F^2(\widetilde{v}) + (2-\alpha)F(\widetilde{v}) + \alpha(\underline{\omega}_H - \widetilde{v})f(\widetilde{v}) = 1-\alpha$$

Observe that only if $H(1-\alpha) > H(q_0)$, then $h(q^*)$ is positive.

$$-\alpha\underline{\omega}_H + \underline{\omega}_L \geq (q_0 - 1)F^{-1}(q_0) + \underline{\omega}_L = (F(r^*) - 1)r^* + \underline{\omega}_L$$

This is equivalent to $\alpha\underline{\omega}_H \leq (1 - F(r^*))r^*$. If this fails, the optimal reserve r^* is above the ironed region, and so a second price auction is optimal.

Finally, observe that

$$h(q^*) \leq h(\beta) = \overline{\omega}_L \leq \underline{\omega}_H - \frac{1 - F_\varepsilon(\underline{\omega}_H)}{f_\varepsilon(\underline{\omega}_H)}$$

which shows that $G(\cdot)$ is convex and completes the proof.

The Complete Information Environment

We show that for almost all realizations of the type vector v, all trembling hand perfect equilibria of a BIN-TAC auction under complete information lead to the same revenues and consumer surplus. Fix the parameters (p, d, r) and let the fixed types $v_1, v_2 \ldots v_N$ wlog be ordered so that v_1 is the highest type, v_2 the second highest etc. We construct a trembling hand perfect equilibrium according to the following algorithm. For every type, compare the payoff to taking the BIN option and the TAC option, assuming all other bidders TAC, and return the higher payoff action. Except on a measure zero set of valuation realizations the agents have strict preferences over these options, so this algorithm terminates in a unique prediction with probability one. Now if some type v_j wants to BIN, it must be that every higher type v_i, $i < j$ also wants to BIN. To see this, notice that the payoff differential from taking BIN is $v_i - v_j$ (all other agents TAC); whereas payoff increase from taking TAC is smaller, no more than $\frac{1}{d}(v_i - v_j)$. So all the action vectors are of the form BIN for the top k agents ($k \geq 0$) and TAC for the rest. Suppose $k = 0$, so that no agent BIN. Then this action vector is the unique equilibrium, as the algorithm solved the correct decision problem for every agent (what to do when all other agents TAC). Suppose then that $k > 0$ and consider the k-th highest agent. He is indifferent between BIN and TAC (he will lose in either case). But his decision is trembling-hand perfect: the only case in which he might get positive surplus is if all other agents TAC, and then the algorithm outputted the correct action for this case. Since this action vector is the unique mutual best response vector in the only case of interest, it follows that this is the unique trembling-hand perfect equilibrium.