Optimal Target Criteria for Stabilization Policy: Technical Appendix [Complete]

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A Proof of Lemmas and Propositions

A.1 Proof of Proposition 1

Here we show that there exists a unique bounded solution to the system of equations consisting of the linearized structural equations (2.8)–(2.10) together with the linearized FOCs (2.16)–(2.17). If we adjoin to these equations the identities

$$\tilde{y}_t = \tilde{y}_t, \tag{A.1}$$

$$E_t\tilde{\varphi}_{t+1} = E_t\tilde{\varphi}_{t+1}, \tag{A.2}$$

then the system consisting of (2.8), (2.16), and (A.1)–(A.2) can be rewritten in matrix form as

$$\bar{M}E_t d_{t+1} = \bar{N}d_t - \bar{N}_s\bar{s}_t, \tag{A.3}$$

where d_t is the $2(m+n)$-dimensional vector

$$d_t \equiv \begin{bmatrix} \tilde{\varphi}_t \\ \tilde{y}_{t-1} \\ E_t\tilde{\varphi}_{t+1} \\ \tilde{y}_t, \end{bmatrix},$$

$\bar{s}_t$ is a vector of exogenous disturbances that includes the elements of both $\tilde{\xi}_t$ and $\tilde{\xi}_{t-1}$, and

$$\bar{M} \equiv \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ I_{m+n} & 0 \end{bmatrix}, \qquad \bar{N} \equiv \begin{bmatrix} -\beta^{-1}\bar{M}_{12}' & 0 \\ 0 & I_{m+n} \end{bmatrix} \tag{A.4}$$

where

$$\bar{M}_{11} \equiv \begin{bmatrix} 0 & \bar{A} \\ \bar{A}' & S \end{bmatrix} = \bar{M}_{11}', \quad \text{and} \quad \bar{M}_{12} \equiv \begin{bmatrix} 0 & -\bar{I} \\ 0 & \beta R' \end{bmatrix}.$$

Here we use the fact that S is symmetric to obtain $\bar{M}_{11} = \bar{M}_{11}'$.

In addition to conditions (A.3), the process $\{d_t\}$ must satisfy (2.9) and (2.17), and thus

$$F_d\left[d_t - E_{t-1}d_t\right] = F_s\left[\bar{s}_t - E_{t-1}\bar{s}_t\right] \tag{A.5}$$

for all $t > t_0$, where F_d is the $(m+n) \times 2(m+n)$ matrix

$$F_d \equiv \begin{bmatrix} S_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{I}_1 \\ 0 & I_m & 0 & 0 \end{bmatrix},$$

using the notation

$$S_2 \equiv [0 \;\; I_{n-k}]$$

for the $(n-k) \times n$ matrix that selects the last $n-k$ elements of any n-vector. (The first $n-k$ rows correspond to conditions (2.17), the next k rows correspond to conditions (2.9), and the final m rows state that the elements of $\tilde{y}_{t-1}$ cannot be affected by surprises in period t.)

In period t_0, the process must satisfy (2.10) and hence

$$F_d\, d_{t_0} = f_{t_0}, \tag{A.6}$$

where f_{t_0} is a vector of $m+n$ initial conditions

$$f_{t_0} \equiv \begin{bmatrix} \tilde{\Theta}_{t_0-1} \\ \bar{A}_1 \tilde{y}_{t_0-1} + \bar{D}\tilde{\xi}_{t_0} - \beta \bar{F}_{t_0} \\ \tilde{y}_{t_0-1} \end{bmatrix},$$

all of which are either predetermined or exogenous.

The following lemmas establish useful properties of the matrix pencil $\bar{M} - \mu \bar{N}$.

Lemma 7 *Given Assumptions 2(b) and 3, the matrix pencil $\bar{M} - \mu\bar{N}$ is regular; that is, its determinant is non-zero for at least some complex μ.*

Proof. The determinant of $\bar{M} - \mu\bar{N}$ can be expressed as follows:

$$\begin{aligned} \det\left(\bar{M} - \mu\bar{N}\right) &= \det \begin{bmatrix} \bar{M}_{11} + \mu\beta^{-1}\bar{M}_{12}' & \bar{M}_{12} \\ I & -\mu I \end{bmatrix} \\ &= \det(-\mu I) \cdot \det\left[\left(\bar{M}_{11} + \mu\beta^{-1}\bar{M}_{12}'\right) - \bar{M}_{12}(-\mu I)^{-1}\right] \\ &= \mu(-1)^{n+m} \cdot \det\left[\bar{M}_{11} + \mu\beta^{-1}\bar{M}_{12}' + \mu^{-1}\bar{M}_{12}\right] \\ &= \mu(-1)^{n+m} \cdot \det \begin{bmatrix} 0 & \bar{A} - \mu^{-1}\bar{I} \\ \bar{A}' - \mu\beta^{-1}\bar{I}' & S + \mu^{-1}\beta R' + \mu R \end{bmatrix}. \end{aligned} \tag{A.7}$$

The matrix pencil $\bar{M} - \mu\bar{N}$ is regular provided that its determinant is non-zero for at least some complex μ.

Suppose the determinant is instead zero for all μ. This means that there must exist finite-order vector polynomials $(\varphi(\mu), y(\mu))$ such that

$$\begin{bmatrix} 0 & \bar{A} - \mu^{-1}\bar{I} \\ \bar{A}' - \mu\beta^{-1}\bar{I}' & S + \mu^{-1}\beta R' + \mu R \end{bmatrix} \begin{bmatrix} \varphi(\mu) \\ y(\mu) \end{bmatrix} = 0 \tag{A.8}$$

for all $\mu \neq 0$, and $(\varphi(\mu), y(\mu))$ are not both equal to zero for all μ. In addition, the solution cannot involve $y(\mu) = 0$. For if there exists a function $\varphi(\mu) \equiv \sum_{i=0}^{k} \varphi_i \mu^i$ satisfying (A.8) with $y(\mu) = 0$, one must have $\left[\bar{A}' - \mu\beta^{-1}\bar{I}'\right]\varphi(\mu) = 0$. But this would imply that the function $\varphi(\mu) \equiv \sum_{i=0}^{k} \varphi_{k-i}\mu^i$ must satisfy (2.13), violating Assumption 2(b). Hence we must have $y(\mu) \neq 0$. Writing $y(\mu) \equiv \sum_{i=0}^{\infty} y_i \mu^i$ (where all but a finite number of the y_i are zero), the first line of (A.8) implies that the sequence $\{y_i\}$ satisfies the hypotheses of Assumption 3. The second line implies that

$$y\left(\mu^{-1}\beta\right)'\left[S + \mu^{-1}\beta R' + \mu R\right] y(\mu) = 0$$

for all μ. Writing this expression in the form $\sum_{j=-(k+1)}^{k+1} \gamma_j \mu^j$, where k is the order of $y(\mu)$, it follows that we must have $\gamma_j = 0$ for all j. In particular, we must have $\gamma_0 = 0$. But γ_0 is just the left-hand side of (2.19), so this violates Assumption 3. It follows that $\det\left(\bar{M} - \mu\bar{N}\right)$ cannot be zero for all μ. ■

It is then possible to factor the polynomial $\det[\lambda\bar{M} - \rho\bar{N}]$ as

$$\prod_{i=1}^{2(n+m)} (\alpha_i\lambda - \beta_i\rho), \tag{A.9}$$

where for any i, the complex numbers α_i and β_i are not both equal to zero. Let s be the number of factors for which $\alpha_i \neq 0$ and $|\beta_i/\alpha_i| < 1$. There must then be $2m + 2n - s$ factors for which $\beta_i \neq 0$ and $|\alpha_i/\beta_i| \leq 1$.

This implies that the matrices $\bar{M}$ and $\bar{N}$ can be decomposed as stated in the following lemma.

Lemma 8 *Given Assumptions 2(b) and 3, there must exist non-singular* $2(m+n)\times 2(m+n)$ *real matrices* $\bar{U}, \bar{V}$ *such that*

$$\bar{U}\bar{M}\bar{V} = \begin{bmatrix} I_s & 0 \\ 0 & \Omega \end{bmatrix}, \qquad \bar{U}\bar{N}\bar{V} = \begin{bmatrix} \Lambda' & 0 \\ 0 & I_{2n+2m-s} \end{bmatrix}. \tag{A.10}$$

Here Ω *is a* $(2m+2n-s)\times(2m+2n-s)$ *real matrix for which all eigenvalues have modulus less than or equal to* 1 *while* Λ *is an* $s \times s$ *real matrix for which all eigenvalues have modulus less than* 1.

Proof. Under Assumptions 2(b) and 3, Lemma 7 implies that $\bar{M} - \mu\bar{N}$ is a real regular matrix pencil of dimensions $(2m + 2n)\times(2m + 2n)$. It follows from Theorem 3 of Gantmacher (1959, Chap. 12), or its version for a real canonical form proved in Appendix D, that there exist real invertible matrices $\tilde{U}, \bar{V}$ of dimensions $(2m + 2n) \times (2m + 2n)$ such that

$$\tilde{U}\bar{M}\bar{V} = \begin{bmatrix} I & 0 \\ 0 & \tilde{G} \end{bmatrix}, \qquad \tilde{U}\bar{N}\bar{V} = \begin{bmatrix} \tilde{H} & 0 \\ 0 & I \end{bmatrix} \tag{A.11}$$

where $\tilde{G}$ is an invertible matrix of the real Jordan form and $\tilde{H}$ is a real nilpotent matrix of the Jordan form.

Let us factor the polynomial $\det\left[\lambda\bar{M} - \rho\bar{N}\right]$ as in (A.9) and let ν $(0 \leq \nu \leq 2m + 2n)$ be the number of factors $(\alpha_i\lambda - \beta_i\rho)$ for which the complex numbers $\beta_i = 0$, while the numbers α_i are necessarily nonzero. (Note that since the eigenvalues of $\bar{M} - \mu\bar{N}$ are the quantities α_i/β_i, these ν factors correspond to the ν "infinite" eigenvalues of $\bar{M} - \mu\bar{N}$.) The existence of a decomposition of the form (A.11) implies that the factors of the characteristic polynomial $\det\left[\lambda\bar{M} - \rho\bar{N}\right]$ in (A.9) are the same as those of

$$\det\left[\lambda I - \rho\tilde{H}\right] \cdot \det\left[\lambda\tilde{G} - \rho I\right].$$

Since $\tilde{H}$ is nilpotent, $\det\left[\lambda I - \rho\tilde{H}\right]$ must correspond to the ν factors for which the $\beta_i = 0$ and $\alpha_i \neq 0$. The matrix pencil $\lambda I - \rho\tilde{H}$ is thus of dimensions $\nu \times \nu$. This implies that the matrix pencil $\lambda\tilde{G} - \rho I$ is of dimensions $(2m + 2n - \nu) \times (2m + 2n - \nu)$ and its determinant is the product of the $2m + 2n - \nu$ factors $(\alpha_i\lambda - \beta_i\rho)$ for which the complex numbers $\beta_i \neq 0$.

(The matrix $\tilde{G}$ has thus $2m+2n-\nu$ eigenvalues denoted by α_i/β_i, all finite). Among these factors, let there be $2m+2n-s$ of them (with $0 \leq s \leq \nu$) for which $\beta_i \neq 0$ and $|\alpha_i/\beta_i| \leq 1$, so that there are $s-\nu$ factors for which $\beta_i \neq 0$ and $|\alpha_i/\beta_i| > 1$. The latter $s-\nu$ factors necessarily have $\alpha_i \neq 0$ and $|\beta_i/\alpha_i| < 1$.

Recalling that $\tilde{G}$ is in real Jordan from, this implies that it is possible to partition it as

$$\begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix}$$

where $\tilde{G}_1 \in \mathbb{R}^{(s-\nu)\times(s-\nu)}$ is a block-diagonal matrix with eigenvalues satisfying $|\alpha_i/\beta_i| > 1$, $\beta_i \neq 0$, and $\tilde{G}_2 \in \mathbb{R}^{(2m+2n-s)\times(2m+2n-s)}$ is a block-diagonal matrix with eigenvalues satisfying $|\alpha_i/\beta_i| \leq 1$, $\beta_i \neq 0$. Since all eigenvalues of $\tilde{G}_1$ are nonzero, the matrix $\tilde{G}_1$ is non-singular. Combining the $s-\nu$ factors associated with $\tilde{G}_1$ with the ν factors associated with the matrix pencil $\lambda I - \rho\tilde{H}$ constitutes s factors for which $\alpha_i \neq 0$ and $|\beta_i/\alpha_i| < 1$.

It follows that the $2\,(n+m) \times 2\,(n+m)$ real matrix

$$\bar{U} \equiv \begin{bmatrix} I_\nu & 0 & 0 \\ 0 & \tilde{G}_1^{-1} & 0 \\ 0 & 0 & I_{2m+2n-s} \end{bmatrix} \tilde{U}$$

is non-singular and satisfies (A.10), where $\Omega \equiv \tilde{G}_2$ is a $(2m+2n-s)\times(2m+2n-s)$ block-diagonal matrix in real Jordan form with blocks corresponding to the factors of (A.9) for which $|\alpha_i/\beta_i| \leq 1$. This implies that

$$\|\Omega\| \leq 1. \tag{A.12}$$

The matrix $\Lambda' \equiv \begin{bmatrix} \tilde{H} & 0 \\ 0 & \tilde{G}_1^{-1} \end{bmatrix}$ is a $s \times s$ block-diagonal matrix in real Jordan form with ν zero eigenvalues (*i.e.*, the eigenvalues of $\tilde{H}$ corresponding to the ν factors $(\alpha_i\lambda - \beta_i\rho)$ for which $\beta_i = 0$), and another $s-\nu$ eigenvalues corresponding to the roots $\beta_i \neq 0$, $|\beta_i/\alpha_i| < 1$. Thus all s eigenvalues of Λ satisfy $|\beta_i/\alpha_i| < 1$, so that

$$\|\Lambda\| < 1. \tag{A.13}$$

■

Because the inequality (A.13) is strict, there also exist values $\delta > 1$ such that

$$\|\delta\Lambda\| < 1. \tag{A.14}$$

In what follows, we shall consider a value of $\delta > 1$ that is small enough for both (1.13) and (A.14) to hold.

Now let the matrices $\bar{U}, \bar{V}$ be partitioned conformably with the partitions in (A.10):

$$\bar{U} = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix} \begin{matrix} \} \; s \text{ rows} \\ \} \; 2m+2n-s \text{ rows} \end{matrix}, \qquad \bar{V} = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 \end{bmatrix} \tag{A.15}$$

where $\bar{V}_1$ and $\bar{V}_2$ are respectively $2(n+m) \times s$ and $2(n+m) \times (2m+2n-s)$ matrices. It follows from the non-singularity of $\bar{U}$ and $\bar{V}$ that the columns of $\left[\begin{array}{cc} \bar{U}_1' & \bar{U}_2' \end{array} \right]$ form a basis for $\mathbb{R}^{2(n+m)}$, as do the columns of $\left[\begin{array}{cc} \bar{V}_1 & \bar{V}_2 \end{array} \right]$. Hence we can represent d_t as

$$d_t = \left[\begin{array}{cc} \bar{V}_1 & \bar{V}_2 \end{array} \right] \left[\begin{array}{c} \psi_t \\ \phi_t \end{array} \right], \qquad (A.16)$$

where ψ_t is of dimension s and ϕ_t is of dimension $2m+2n-s$. The vectors (ψ_t, ϕ_t) can be uniquely re-constructed from the vector d_t, and vice versa.

The decomposition (A.10) defines stable and unstable subspaces for the matrix pencil $\bar{M} - \mu \bar{N}$. In particular, for any $\delta \geq 1$, let us define the $\delta-$stable subspace $\mathcal{D}_\delta$ as the set of values d_{t_0} for which there exists a deterministic sequence $\{d_t\}$ for $t \geq t_0$ consistent with this value of d_{t_0}, satisfying

$$\bar{M} d_{t+1} = \bar{N} d_t \qquad (A.17)$$

for all $t \geq t_0$, and such that

$$\lim_{t \to \infty} \delta^t d_t = 0. \qquad (A.18)$$

(In the case that $\delta = 1$, we shall call $\mathcal{D} \equiv \mathcal{D}_1$ simply the stable subspace.) We then have the following result regarding the dimension of this linear space.

Lemma 9 *Given Assumptions 2(b) and 3, let $\mathcal{D}_\delta$ be the δ-stable subspace of the matrix pencil $\bar{M} - \mu \bar{N}$ corresponding to a value of δ such that (A.14) holds. Then $\mathcal{D}_\delta$ is a linear space of dimension s, the dimension of the square matrix Λ in (A.10).*

Proof. Under Assumptions 2(b) and 3, Lemma 8 holds, and we can then rewrite (A.10) as:

$$\bar{U}_1 \bar{M} \bar{V}_1 = I_s, \qquad \bar{U}_1 \bar{M} \bar{V}_2 = 0 \qquad (A.19)$$
$$\bar{U}_2 \bar{M} \bar{V}_1 = 0, \qquad \bar{U}_2 \bar{M} \bar{V}_2 = \Omega \qquad (A.20)$$

and

$$\bar{U}_1 \bar{N} \bar{V}_1 = \Lambda', \qquad \bar{U}_1 \bar{N} \bar{V}_2 = 0 \qquad (A.21)$$
$$\bar{U}_2 \bar{N} \bar{V}_1 = 0, \qquad \bar{U}_2 \bar{N} \bar{V}_2 = I_{2m+2n-s}. \qquad (A.22)$$

We observe from these orthogonality relations that the inverse transformations can be written as

$$\bar{U}^{-1} = \left[\begin{array}{cc} \bar{M}\bar{V}_1 & \bar{N}\bar{V}_2 \end{array} \right], \qquad \bar{V}^{-1} = \left[\begin{array}{c} \bar{U}_1 \bar{M} \\ \bar{U}_2 \bar{N} \end{array} \right]. \qquad (A.23)$$

(Here we use the fact that because $\bar{U}$ and $\bar{V}$ are non-singular, we know that unique inverses exist.)

We can then pre-multiply the equations in (A.10) by $\bar{U}^{-1}$, using (A.23), to obtain:

$$\bar{M} \bar{V}_2 = \bar{N} \bar{V}_2 \Omega \qquad (A.24)$$
$$\bar{M} \bar{V}_1 \Lambda' = \bar{N} \bar{V}_1. \qquad (A.25)$$

We can similarly post-multiply the equations in (A.10) by $\bar{V}^{-1}$, using (A.23), to obtain:

$$\begin{aligned} \Lambda' \bar{U}_1 \bar{M} &= \bar{U}_1 \bar{N} && \text{(A.26)} \\ \bar{U}_2 \bar{M} &= \Omega \bar{U}_2 \bar{N}. && \text{(A.27)} \end{aligned}$$

Because $\bar{U}^{-1}$ and $\bar{V}^{-1}$ must be non-singular matrices, we observe from (A.23) that $\bar{M}\bar{V}_1$, $\bar{N}\bar{V}_2$, $\bar{M}'\bar{U}_1'$, and $\bar{N}'\bar{U}_2'$ must each be matrices of full rank.

Pre-multiplying (A.17) by $\bar{U}_2$ and using (A.27), we obtain

$$\Omega \bar{U}_2 \bar{N} d_{t+1} = \bar{U}_2 \bar{N} d_t$$

for each $t \geq t_0$. Then using (A.16) to substitute for d_t on both sides of this equation, and using (A.22), we obtain

$$\Omega \phi_{t+1} = \phi_t,$$

which in turn implies that

$$\delta^t \phi_t = (\delta^{-1}\Omega)\delta^{t+1}\phi_{t+1} \qquad \text{(A.28)}$$

for each $t \geq t_0$. Repeated application of (A.28) implies that

$$\delta^t \phi_t = (\delta^{-1}\Omega)^k \delta^{t+k}\phi_{t+k} \qquad \text{(A.29)}$$

for arbitrary $k \geq 1$. Then in the case of any sequence $\{d_t\}$ satisfying (A.18), (A.12) implies that the right-hand side of (A.29) converges to zero for large k. Hence we must have $\phi_t = 0$ for all $t \geq t_0$ in the case of any such sequence. Thus d_t must be a vector of the form $d_t = \bar{V}_1 \psi_t$ for all t.

Pre-multiplying (A.17) by $\bar{U}_1$ and again using (A.16) to substitute for d_t, one can similarly show that

$$\psi_{t+1} = \Lambda' \psi_t$$

for all $t \geq t_0$. Given a vector ψ_{t_0}, this law of motion can be solved for the complete sequence $\{\psi_t\}$ and hence for the implied sequence $\{d_t\}$. Since

$$\delta^t \psi_t = (\delta \Lambda')^{t-t_0} \delta^{t_0} \psi_{t_0}$$

for any t, it follows from (A.14) that (A.18) must be satisfied. Hence the $\delta-$stable subspace $\mathcal{D}_\delta$ consists of all vectors of the form $d_{t_0} = \bar{V}_1 \psi_{t_0}$ for some vector ψ_{t_0}. Since $\bar{V}$ is invertible, this linear space must be of dimension s (the number of columns of $\bar{V}_1$). ■

We turn now to a further characterization of the dimension s. Since $\bar{M} - \mu \bar{N}$ is a regular pencil, a pair (λ, ρ) determines an eigenvalue μ of $\bar{M} - \mu \bar{N}$ if $\det[\lambda \bar{M} - \rho \bar{N}] = 0$ and $\rho - \lambda \mu = 0$. (In particular, a pair (λ, ρ) determines an infinite eigenvalue of $\bar{M} - \mu \bar{N}$ if $\det[\lambda \bar{M} - \rho \bar{N}] = 0$, and $\rho \neq 0, \lambda = 0$.) Because of the symmetries in the elements of the matrices $\bar{M}$ and $\bar{N}$, the eigenvalues of the pencil $\bar{M} - \mu \bar{N}$ also satisfy the following symmetry.[23]

[23] This demonstration that the eigenvalues come in "reciprocal pairs" extends to our environment a standard result in the theory of linear-quadratic optimal control (e.g., Hansen and Sargent, 2010, chap. 8).

Lemma 10 *Given Assumptions 2(b) and 3, the set of values of* $\mu = \rho/\lambda$ *for which* $\det[\lambda\bar{M} - \rho\bar{N}] = 0$ *is such that if* μ *belongs to the set, so do the numbers* $\beta\mu^{-1}$*, and the complex conjugates* $\bar{\mu}$ *and* $\overline{\beta\mu^{-1}}$*. In particular, if the equation holds for* $\rho = 0$ *(and arbitrary* λ*), then it also holds for* $\lambda = 0$ *(and arbitrary* ρ*).*

Proof. Given Assumptions 2(b) and 3, Lemma 7 implies that the matrix pencil $\bar{M} - \mu\bar{N}$ is regular. Hence the matrix pencil $\bar{M} - \mu\hat{N}$ where $\hat{N} \equiv \beta^{1/2}\bar{N}$ is also regular. Let us define the $2(n+m) \times 2(n+m)$ matrix

$$J \equiv \begin{bmatrix} 0 & I_{n+m} \\ -I_{n+m} & 0 \end{bmatrix},$$

and observe that

$$\bar{M}'J\bar{M} = \hat{N}'J\hat{N},$$

so that the transposed matrix pencil $(\bar{M} - \mu\hat{N})'$ is symplectic. It follows that the generalized eigenvalues of the transposed pencil $(\bar{M} - \mu\hat{N})'$ are symmetric with respect to the unit circle (see Theorems 4 and 5 of Pappas, Laub and Sandell, 1980): if $\mu \in \mathbb{C}$ is a generalized eigenvalue of the real matrix pencil $(\bar{M} - \mu\hat{N})'$, then so are μ^{-1} and the complex conjugates $\bar{\mu}$, $\overline{\mu^{-1}}$. In particular, if $\mu = 0$ is an eigenvalue of $(\bar{M} - \mu\hat{N})'$, so is $\mu = \infty$.

Since $\det[\bar{M} - \mu\hat{N}] = \det[\bar{M}' - \mu\hat{N}']$ for all μ, it follows that if $\mu \in \mathbb{C}$ is an eigenvalue of $(\bar{M} - \mu\hat{N})$, then so are μ^{-1} and the complex conjugates $\bar{\mu}$, $\overline{\mu^{-1}}$. Moreover, $\det[\lambda\bar{M} - \rho\hat{N}] = 0$ if and only if $\det[\lambda\bar{M} - \beta^{1/2}\rho\bar{N}] = 0$. Hence μ is a generalized eigenvalue of $(\bar{M} - \mu\bar{N})$ if and only if $\beta^{-1/2}\mu$ is a generalized eigenvalue of the transformed pencil $(\bar{M} - \mu\hat{N})$. It then follows that $\beta\mu^{-1}$, $\bar{\mu}$, and $\overline{\beta\mu^{-1}}$ must also be generalized eigenvalues of $(\bar{M} - \mu\bar{N})$. ■

Lemma 11 *Given Assumptions 1(b), 2(b) and 3, the dimension of the square matrix* Λ *in the decomposition (A.10) must be exactly* $s = m + n$*. Hence the matrix pencil* $\bar{M} - \mu\bar{N}$ *has exactly* $m + n$ *generalized eigenvalues satisfying* $|\mu| < \beta$ *and another* $m + n$ *generalized eigenvalues (some of which may be infinite) satisfying* $|\mu| > 1$*, and the stable subspace* $\mathcal{D}$ *is of dimension* $m + n$*. The dimension of the square matrix* Ω *is also* $m + n$*, and this matrix satisfies*

$$\|\Omega\| < \beta. \tag{A.30}$$

Proof. Assumption 1(b) implies that for any initial conditions close enough to consistency with the optimal steady state, there must exist a solution to the first-order conditions (for the deterministic case in which $\xi_t = \bar{\xi}$ at all times) in which (1.13) holds. It follows that for arbitrary initial conditions f_{t_0}, there must be a sequence $\{d_t\}$ satisfying the linearized FOCs (A.17) for all $t \geq t_0$, such that d_{t_0} is consistent with (A.6), and such that $\{\delta^t d_t\}$ is a bounded sequence.[24] It is then furthermore possible to choose a $\delta > 1$ (possibly slightly smaller than the δ referred to in Assumption 1(b)) such that (A.18) is satisfied. For a small enough choice of $\delta > 1$, (A.14) must hold as well. Hence there must exist $\delta > 1$ for which (A.14) holds, and

[24] Note that convergence in the exact nonlinear dynamics only implies that the sequence must not explode in the linearized dynamics, since the rate of convergence might asymptotically decrease to zero.

such that for any initial conditions f_{t_0}, there exists a vector d_{t_0} in the $\delta-$stable subspace $\mathcal{D}_\delta$ consistent with (A.6).

It follows from our characterization of the $\delta-$stable subspace in the proof of Lemma 9 that there must exist a vector ψ_{t_0} such that

$$[F_d \bar{V}_1]\ \psi_{t_0} = f_{t_0}. \tag{A.31}$$

It is easily seen that any values for the $m+n$ elements of f_{t_0} can be arranged through a suitable specification of the $n-k$ elements of $\tilde{\Theta}_{t_0-1}$, the k elements of $\bar{F}_{t_0}$, and the m elements of $\tilde{y}_{t_0-1}$. Hence the right-hand side of (A.31) can be any element of $\mathbf{R}^{m+n}$. Then in order for a solution to exist for arbitrary initial conditions, it is necessary that

$$\text{rank } F_d \bar{V}_1 = m + n. \tag{A.32}$$

This requires that $s \geq m + n$.

We further note that the decomposition (A.10) implies that the generalized eigenvalues of the pencil $\bar{M} - \mu \bar{N}$ consist of the $2m + 2n - s$ eigenvalues μ_i of the matrix Ω and the reciprocals of the s eigenvalues λ_j of the matrix Λ. Lemma 10 implies that for each eigenvalue λ_j of Λ, $\beta\lambda_j$ must also be a generalized eigenvalue of the pencil $\bar{M} - \mu\bar{N}$; and since $|\lambda_j| < 1$, this must be a generalized eigenvalue with modulus less than β, and therefore an eigenvalue of Ω rather than the reciprocal of any eigenvalue of Λ. Hence for each eigenvalue λ_j of Λ, $\beta\lambda_j$ must be an eigenvalue of Ω. This requires that Ω be of at least the dimension of Λ, and hence that $s \leq m + n$. Therefore $s = m + n$ exactly. The matrix Ω is of dimension $m + n$, and its eigenvalues all satisfy $|\mu| < \beta$, which implies (A.30).

Finally, it follows that $[F_d \bar{V}_1]$ must be a non-singular square matrix, so that (A.31) can be solved for ψ_{t_0} for any specification of the initial conditions f_{t_0}. Since the largest eigenvalue of Λ must have a modulus strictly less than 1, any initial condition of the form $d_{t_0} = \bar{V}_1 \psi_{t_0}$ gives rise to a sequence $\{d_t\}$ satisfying (A.18) for $\delta = 1$. Hence this linear space of dimension $m + n$ corresponds to the stable subspace. ■

In the proof of Lemma 11, it has already been established that for any initial conditions f_{t_0}, there exists a deterministic solution $\{d_t\}$ to the linearized FOCs that converges exponentially to the steady state for large t. This result can then be directly extended to the case of bounded fluctuations in the exogenous disturbances $\{\tilde{\xi}_t\}$, yielding the result stated in the proposition.

Given a bounded stochastic process $\{\tilde{\xi}_t\}$ for the exogenous disturbances and a vector f_{t_0} of initial conditions, we are interested in stochastic processes $\{d_t\}$ such that (i) $\{d_t\}$ is bounded; (ii) (A.3) is satisfied for all $t \geq t_0$; (iii) (A.5) is satisfied for all $t > t_0$; and (iv) d_{t_0} satisfies (A.6). Pre-multiplying (A.3) by $\bar{U}_2$, we can show as in the proof of Lemma 9 that

$$\Omega \bar{U}_2 \bar{N} E_t d_{t+1} = \bar{U}_2 \bar{N} d_t - \bar{U}_2 \bar{N}_s \bar{s}_t,$$

or equivalently that

$$E_t[(I - \Omega L^{-1}) \bar{U}_2 \bar{N} d_t] = \bar{U}_2 \bar{N}_s \bar{s}_t.$$

Using (A.16) to substitute for d_t, this can alternatively be written

$$E_t[(I - \Omega L^{-1}) \phi_t] = \bar{U}_2 \bar{N}_s \bar{s}_t. \tag{A.33}$$

Because of (A.30), the operator $I-\Omega L^{-1}$ is invertible on the linear space of bounded processes $\{\phi_t\}$, so that for any disturbance process such that $||\bar{s}|| < \infty$, (A.33) has a unique solution such that $||\phi|| < \infty$, given by

$$\phi_t = E_t[(I - \Omega L^{-1})^{-1}\bar{U}_2\bar{N}_s\bar{s}_t]. \tag{A.34}$$

Similarly, pre-multiplying (A.3) by $\bar{U}_1$ and using (A.16) to substitute for d_t yields

$$E_t\psi_{t+1} = \Lambda'\psi_t - \bar{U}_1\bar{N}_s\bar{s}_t. \tag{A.35}$$

Using (A.16) to substitute for d_t in (A.5), and shifting the time index by one period, yields

$$F_d\bar{V}_1\left[\psi_{t+1} - E_t\psi_{t+1}\right] = F_s\left[\bar{s}_t - E_{t-1}\bar{s}_t\right] - F_d\bar{V}_2\left[\phi_{t+1} - E_t\phi_{t+1}\right]$$

for each $t \geq t_0$. Since $F_d\bar{V}_1$ is an invertible square matrix (as shown in the proof of Lemma 11), this can be solved uniquely for ψ_{t+1}. Substituting expression (A.35) for the conditional expectation $E_t\psi_{t+1}$ in this equation, and the solution (A.34) for both ϕ_{t+1} and its conditional expectation, we obtain a law of motion of the form

$$\psi_{t+1} = \Lambda'\psi_t + r_{t+1} \tag{A.36}$$

for all $t \geq t_0$, where $\{r_t\}$ is a process satisfying $||r|| < \infty$ that has been uniquely determined as a function of the evolution of the exogenous disturbances.

Finally, using (A.16) to substitute for d_{t_0} in (A.6) we obtain

$$F_d\bar{V}_1\psi_{t_0} = f_{t_0} - F_d\bar{V}_2\phi_{t_0}.$$

Using the solution (A.34) to substitute for ϕ_{t_0} in this equation, the invertibility of $F_d\bar{V}_1$ implies that this equation has a unique solution for ψ_{t_0} for any specification of the initial conditions f_{t_0} and the process for the exogenous disturbances. Given this initial condition for ψ_{t_0}, the law of motion (A.36) can then be integrated forward, yielding a unique solution for the evolution of $\{\psi_t\}$ for all $t \geq t_0$. It follows from (A.14) and the fact that $||r|| < \infty$ that this solution will satisfy $||\psi|| < \infty$. Our solutions for the processes $\{\phi_t, \psi_t\}$ then imply a unique solution for the process $\{d_t\}$, using (A.16), and the bounds satisfied by the two solutions imply that $||d|| < \infty$ as well. Hence there is a unique solution satisfying this bound. QED.

A.2 Proof of Lemma 1

There is no vector $\hat{\varphi}_2(\mu) \neq 0$ such that $\hat{\varphi}_2(\mu)'\left[J_2 - \mu B_2\right] = 0$ for all μ. For if there did, the vector

$$\varphi(\mu) = P^{-1}\begin{bmatrix} 0 \\ \hat{\varphi}_2(\mu) \end{bmatrix} \neq 0$$

would satisfy (2.13), and this would violate Assumption 2. This implies that the pencil $J_2 - \mu B_2$ must be of rank q, which (since it is a square pencil of dimension $q \times q$) implies that it is a regular pencil.

It follows from Theorem 3 of Gantmacher (1959, Chap. 12), or its version for a real canonical form proved in Appendix D, that $J_2 - \mu B_2$ can be reduced to a strictly equivalent pencil of the form

$$\begin{bmatrix} I_l - \mu \hat{G}' & 0 \\ 0 & \hat{H}' - \mu I_{q-l} \end{bmatrix} \tag{A.37}$$

where $\hat{G}' \in \mathbb{R}^{l \times l}$ $(0 \leq l \leq q)$ is a nilpotent matrix of the Jordan form (*i.e.*, with ones on the first super diagonal and zeros everywhere else), and $\hat{H}' \in \mathbb{R}^{(q-l)\times(q-l)}$ is a block-diagonal matrix of the real Jordan form. We can without loss of generality arrange the Jordan blocks of $\hat{H}'$ as

$$\hat{H}' = \begin{bmatrix} \hat{H}'_{11} & 0 \\ 0 & H' \end{bmatrix}$$

where the invertible matrix $\hat{H}_{11}$ contains the eigenvalues with modulus greater than or equal to β, and H contains only eigenvalues with modulus less than β. Premultiplying the pencil (A.37) by the invertible block-diagonal matrix

$$\begin{bmatrix} I_l & 0 & 0 \\ 0 & (\hat{H}'_{11})^{-1} & 0 \\ 0 & 0 & I_{q-k_2} \end{bmatrix}$$

yields a strictly equivalent matrix pencil of the form (2.29), where all eigenvalues of

$$G' = \begin{bmatrix} \hat{G}' & 0 \\ 0 & (\hat{H}'_{11})^{-1} \end{bmatrix}.$$

have modulus less than or equal to β^{-1} and all eigenvalues of H have modulus less than β. It remains only to determine the dimensions of G and H.

The existence of a decomposition of the form (2.29) implies that the factors of the characteristic polynomial $P(\lambda, \mu)$ defined in Assumption 4 are the same as those of

$$\det[\lambda I - \mu G'] \cdot \det[\lambda H' - \mu I].$$

This implies the existence of a factorization of the form (2.22), where the $\{\gamma_i\}$ are the eigenvalues of G and the $\{\eta_j\}$ are the eigenvalues of H. It follows that G must be of dimension $k_2 \times k_2$ and H must be of dimension $(q-k_2)\times(q-k_2)$. Finally, since by Assumption 4(b), $|\gamma_i| < 1$ for all i, all eigenvalues of G must have modulus strictly smaller than 1.

A.3 Proof of Proposition 2

Premultiplying (2.24) by N_2' yields the pair of equations

$$E_t x_{1,t+1} = G' x_{1,t} + N'_{21}\Gamma_2 \tilde{\xi}_t \tag{A.38}$$

$$H' E_t x_{2,t+1} = x_{2,t} + N'_{22}\Gamma_2 \tilde{\xi}_t \tag{A.39}$$

using the decomposition (2.29), defining

$$x_t \equiv \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \equiv \left(R_2^{-1}\right)' y^*_{2t}, \tag{A.40}$$

and partitioning $N_2' \equiv \begin{bmatrix} N_{21}' \\ N_{22}' \end{bmatrix}$ conformably with the partition of x_t.

Because all eigenvalues of H' have modulus less than β, and $\{\tilde{\xi}_t\}$ is bounded for all dates $t \geq t_0 - 1$, there is a unique process $\{x_{2t}\}$ consistent with (A.39) and such that $||x_2|| < \infty$, namely

$$x_{2t} = -\sum_{j=0}^{\infty} (H')^j N_{22}' \Gamma_2 E_t \tilde{\xi}_{t+j}. \tag{A.41}$$

Then in any period t, given the past, current and expected future values of the exogenous disturbance process, and given the lagged expectations of $J_2 E_{t-1} y_{2t}^*$, equation (2.26) determines the value of $X_1' J_2 y_{2,t}^*$, while equation (A.41) determines the value of x_{2t} and hence of $H' x_{2t}$. We then have a system of equations of the form

$$\begin{bmatrix} X_1' \\ \begin{bmatrix} 0 & I_{q-k_2} \end{bmatrix} N_2' \end{bmatrix} J_2 y_{2,t}^* = \begin{bmatrix} X_1' J_2 y_{2,t}^* \\ H' x_{2t} \end{bmatrix}$$

to solve for $J_2 y_{2,t}^*$, where all elements of the matrix on the right-hand side have been computed. Since the matrix on the left-hand side is invertible by Assumption 5, this system has a unique solution for $J_2 y_{2,t}^*$. Using this solution, we can in turn solve for

$$x_{1,t} = [I \quad 0] \; N_2' J_2 y_{2,t}^*.$$

Combining this solution for x_{1t} with (A.41), we have a unique solution for the entire vector x_t, given values of $J_2 E_{t-1} y_{2,t}^*$ and the evolution of the exogenous disturbances. And given the value of x_t in any period, equations (A.38) and (A.39) uniquely determine the values of $E_t [x_{1,t+1}]$ and $E_t [H' x_{2,t+1}]$ respectively. This allows us to uniquely determine

$$J_2 E_t y_{2,t+1}^* = N_2^{-1} \begin{bmatrix} E_t x_{1,t+1} \\ E_t [H' x_{2,t+1}] \end{bmatrix}.$$

Thus starting from given initial conditions $J_2 E_{t_0 - 1} y_{2,t_0}^*$, we can uniquely solve for x_{t_0}, use this to uniquely solve for $J_2 E_{t_0} y_{2,t_0+1}^*$, use this to uniquely solve for x_{t_0+1}, and so on recursively, eventually obtaining a unique solution for the entire process $\{x_t\}$, and hence a unique solution for the entire process $\{y_{2t}^*\}$, using the relation $y_{2t}^* = R_2' x_t$.

This solution $\{y_{2t}^*\}$ is the only solution such that $||y_2^*|| < \infty$, if any solution exists. But one easily verifies that it is indeed such a solution. By construction, (2.26) is satisfied each period, and also both (A.38) and (A.39), which suffice to imply that (2.24) is satisfied each period. Moreover, the fact that all eigenvalues of G have modulus less than 1 implies that the process $\{x_t\}$ constructed in this way satisfies $||x|| < \infty$, so that the associated process $\{y_{2t}^*\}$ satisfies $||y_2^*|| < \infty$. Hence all conditions for a solution are satisfied.

A.4 Proof of Lemma 2

Using the definition of q_t and the fact that the matrix W is symmetric, we may rewrite the objective function (3.3) as

$$\pi (y_t; \xi_t) = -\frac{1}{2} \left[y_t' w W w' y_t - 2 q^* (\xi_t)' W w' y_t + q^* (\xi_t)' W q^* (\xi_t) \right]$$

so that

$$
\begin{aligned}
D_1\pi\left(y_t;\xi_t\right) &= -\left[y_t'w - q^*\left(\xi_t\right)'\right]Ww' \qquad \text{(A.42)} \\
D_1\left[\left(D_1\pi\left(y_t;\xi_t\right)\right)'\right] &= -wWw' \\
D_2\left[\left(D_1\pi\left(y_t;\xi_t\right)\right)'\right] &= wW\left[Dq^*\left(\xi_t\right)\right].
\end{aligned}
$$

The fact that the targets $q^*\left(\xi_t\right)$ are achievable implies that in steady state, $D_1\pi\left(\bar{y},\bar{\xi}\right)=0$, using (3.4) and (A.42). It then follows from (1.12) that

$$
\bar{\varphi}'\left(\bar{I}-\beta\bar{A}\right)=0.
$$

Assumption 2(b) then implies that $\bar{\varphi}=0$, so that S, R and $B\left(L\right)$ reduce to

$$
S=-wWw', \quad R=0, \quad B\left(L\right)=wW\left[Dq^*\left(\bar{\xi}\right)\right]\cdot L.
$$

The target variables and target values are then given by

$$
\begin{aligned}
\tau_t &= wWw'\tilde{y}_t = wW\left(q_t-\bar{q}\right), \\
\tau_t^* &= wW\left[Dq^*\left(\bar{\xi}\right)\right]\cdot\tilde{\xi}_t,
\end{aligned}
$$

where $\bar{q}\equiv w'\bar{y}$. Performing a first-order approximation to $q^*\left(\xi_t\right)$, we obtain

$$
q_t^*\equiv q^*\left(\xi_t\right)=q^*\left(\bar{\xi}\right)+\left[Dq^*\left(\bar{\xi}\right)\right]\cdot\tilde{\xi}_t+\mathcal{O}(\epsilon^2)
$$

so that, using (3.4), we have

$$
\left[Dq^*\left(\bar{\xi}\right)\right]\cdot\tilde{\xi}_t=q_t^*-\bar{q}+\mathcal{O}(\epsilon^2).
$$

Using this, we can express the "target gaps" $\tau_t-\tau_t^*$ as

$$
\tau_t-\tau_t^*=wW\left(q_t-\bar{q}\right)+wW\left(q_t^*-\bar{q}\right)+\mathcal{O}(\epsilon^2)=wW\left(q_t-q_t^*\right)+\mathcal{O}(\epsilon^2).
$$

A.5 Proof of Lemma 3

The fact that the pencil $\bar{A}-\mu\bar{I}$ is of rank $n<m$ implies that the columns are linearly dependent, *i.e.*, that there exists $y\left(\mu\right)$ such that

$$
\left[\bar{A}-\mu\bar{I}\right]y\left(\mu\right)=0 \qquad \text{(A.43)}
$$

for all μ, though by Assumption 2, $y\left(\mu\right)$ is of order greater than zero. Let $\epsilon_1\geq 1$ be the minimal order of solution $y\left(\mu\right)$ that exists to (A.43). (A solution of finite order ϵ_1 necessarily exists.) Then Theorem 4 of Gantmacher (1959, chap. 12) implies that the pencil $\bar{A}-\mu\bar{I}$ is strictly equivalent to a pencil of the form

$$
\left[\begin{array}{cc} M_1\left(\mu\right)' & 0 \\ 0 & B_2^{(1)}-\mu J_2^{(1)} \end{array}\right]
$$

where $B_2^{(1)}-\mu J_2^{(1)}$ is a pencil for which the equation

$$
[B_2^{(1)}-\mu J_2^{(1)}]\ \hat{y}^{(1)}\left(\mu\right)=0 \qquad \text{(A.44)}
$$

has no solution of order less than ϵ_1.

If (A.44) nonetheless has a nonzero solution of minimal order $\epsilon_2 \geq \epsilon_1$, then Theorem 4 of Gantmacher can be applied again to the pencil $B_2^{(1)} - \mu J_2^{(1)}$. Proceeding in this way, one eventually transforms the pencil $\bar{A} - \mu \bar{I}$ into a pencil of the form shown in (3.6), where the sequence of indices satisfies

$$\epsilon_p \geq ... \geq \epsilon_2 \geq \epsilon_1 \geq 1,$$

and $[B_2 - \mu J_2]$ is a pencil for which the equation

$$[B_2 - \mu J_2]\,\hat{y}\,(\mu) = 0$$

has no nonzero solution, *i.e.*, the columns are linearly independent.

By the same argument as in the proof of Lemma 1, Assumption 2 implies that there is also no vector $\hat{\varphi}\,(\mu) \neq 0$ such that

$$\hat{\varphi}\,(\mu)'\,[B_2 - \mu J_2] = 0,$$

i.e., the rows of the pencil are also linearly independent. Hence $[B_2 - \mu J_2]$ must be a square pencil of some dimension $q \times q$. (Note that it is possible that $q = 0$, *i.e.*, that B_2 and J_2 are null matrices.)

Adding up the columns of the matrix in (3.6), one observes that

$$\sum_{i=1}^{p} \epsilon_i + q = n.$$

Adding up the rows, one similarly observes that

$$\sum_{i=1}^{p} (\epsilon_i + 1) + q = m,$$

from which it follows that the number of M_i blocks must equal $p = m - n$.

This theorem implies that there exist nonsingular square matrices P, Q of dimensions $n \times n$ and $m \times m$ respectively that satisfy (3.6).

A.6 Proof of Lemma 4

Suppose that (3.2) holds for any period $t > t_0$, and suppose that Assumption 2 holds. Lemma 3 then implies that there exist matrices P, Q that define the decomposition (3.6), i.e., a decomposition of the form (2.20). Assumption 4 then allows us to decompose conditions (3.2) into separate subsystems as well. It follows from (2.21) that

$$\begin{aligned} \begin{bmatrix} \hat{\varphi}_{1,t} \\ \hat{\varphi}_{2,t} \end{bmatrix} &= P \begin{bmatrix} T_1' & 0 \\ 0 & T_2' \end{bmatrix} \begin{bmatrix} \left(T_1^{-1}\right)' & 0 \\ 0 & \left(T_2^{-1}\right)' \end{bmatrix} \begin{bmatrix} \tilde{\varphi}_{1,t} \\ \tilde{\varphi}_{2,t} \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 & U_2 & 0 \\ 0 & X_1 & 0 & X_2 \end{bmatrix} \begin{bmatrix} \left(T_1^{-1}\right)' \tilde{\varphi}_{1,t} \\ \left(T_2^{-1}\right)' \tilde{\varphi}_{2,t} \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned}\hat{\varphi}_{1,t} &= \begin{bmatrix} U_1 & 0 \end{bmatrix} \left(T_1^{-1}\right)' \tilde{\varphi}_{1,t} + \begin{bmatrix} U_2 & 0 \end{bmatrix} \left(T_2^{-1}\right)' \tilde{\varphi}_{2,t} && \text{(A.45)}\\ \hat{\varphi}_{2,t} &= \begin{bmatrix} 0 & X_1 \end{bmatrix} \left(T_1^{-1}\right)' \tilde{\varphi}_{1,t} + \begin{bmatrix} 0 & X_2 \end{bmatrix} \left(T_2^{-1}\right)' \tilde{\varphi}_{2,t}. && \text{(A.46)}\end{aligned}$$

Equations (A.45) and (A.46) respectively imply that

$$\begin{aligned}\hat{\varphi}_{1,t} - E_{t-1}\hat{\varphi}_{1,t} &= \begin{bmatrix} U_1 & 0 \end{bmatrix} \left(T_1^{-1}\right)' \left(\tilde{\varphi}_{1,t} - E_{t-1}\tilde{\varphi}_{1,t}\right) && \text{(A.47)}\\ \hat{\varphi}_{2,t} - E_{t-1}\hat{\varphi}_{2,t} &= \begin{bmatrix} 0 & X_1 \end{bmatrix} \left(T_1^{-1}\right)' \left(\tilde{\varphi}_{1,t} - E_{t-1}\tilde{\varphi}_{1,t}\right). && \text{(A.48)}\end{aligned}$$

Recall from Assumption 4 that both $\begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$ are invertible matrices, and define the matrices V_1, V_2, W_1, and W_2 as in (3.10). Note that it follows from these definitions that

$$\begin{aligned}V_1U_1 &= I_{k_1}, \quad V_1U_2 = 0, \quad V_2U_1 = 0, \quad V_2U_2 = I_{\tilde{n}-k_1} && \text{(A.49)}\\ W_1X_1 &= I_{k_2}, \quad W_1X_2 = 0, \quad W_2X_1 = 0, \quad W_2X_2 = I_{q-k_2}. && \text{(A.50)}\end{aligned}$$

Premultiplying (A.47) by V_2 yields (3.11); premultiplying (A.48) by W_2 yields (3.12). Hence (3.11)-(3.12) must hold for all $t > t_0$.

Conversely, suppose that (3.11)-(3.12) hold in some period $t > t_0$. Using (A.45), (3.11) implies that

$$\begin{bmatrix} I & 0 \end{bmatrix} \left(T_2^{-1}\right)' \left(\tilde{\varphi}_{2,t} - E_{t-1}\tilde{\varphi}_{2,t}\right) = 0.$$

Similarly, using (A.46), (3.12) implies that

$$\begin{bmatrix} 0 & I \end{bmatrix} \left(T_2^{-1}\right)' \left(\tilde{\varphi}_{2,t} - E_{t-1}\tilde{\varphi}_{2,t}\right) = 0.$$

Together, these conditions imply that

$$\left(T_2^{-1}\right)' \left(\tilde{\varphi}_{2,t} - E_{t-1}\tilde{\varphi}_{2,t}\right) = 0$$

which implies (3.2). Hence (3.2) must hold for all $t > t_0$.

A.7 Proof of Proposition 3

In order to prove Proposition 3, we make use of a further preliminary result.

Lemma 12 *Suppose that there exist matrices P, Q that define a decomposition of the form (2.20), and that Assumptions 2, 4 and 5 hold. Then the $q \times q$ matrix*

$$\begin{bmatrix} W_2 \\ \begin{bmatrix} I_{k_2} & 0 \end{bmatrix} R_2 J_2' \end{bmatrix} \tag{A.51}$$

is invertible.

Proof. Let x be an arbitrary vector of length q, partitioned as in (A.40). By Assumption 5, knowing the values of the vectors $X_1' J_2 R_2' x$ and $[0 \ I] \ N_2' J_2 R_2' x$ allows one to reconstruct the entire vector $J_2 R_2' x$, and hence all elements of

$$N_2' J_2 R_2' x = \begin{bmatrix} x_1 \\ H' x_2 \end{bmatrix}.$$

Since the elements of $[0 \ I] \ N_2' J_2 R_2' x = H' x_2$ provide information only about the elements of x_2, it must be that each of the k_2 independent directions of variation of the elements of x_1 affects the elements of $X_1' J_2 R_2' x$ in an independent direction. Thus Assumption 5 implies that the $k_2 \times k_2$ matrix

$$\Xi \equiv X_1' J_2 R_2' \begin{bmatrix} I_{k_2} \\ 0 \end{bmatrix} \tag{A.52}$$

is invertible.

Next, let $\hat{\varphi}$ be another arbitrary vector of length q, and let

$$\check{\varphi} \equiv N_2^{-1} \hat{\varphi} \equiv \begin{bmatrix} \check{\varphi}_1 \\ \check{\varphi}_2 \end{bmatrix}.$$

One observes that

$$R_2 J_2' \hat{\varphi} = [R_2 J_2' N_2] \, \check{\varphi} = \begin{bmatrix} I & 0 \\ 0 & H \end{bmatrix} \check{\varphi} = \begin{bmatrix} \check{\varphi}_1 \\ H \check{\varphi}_2 \end{bmatrix},$$

Then in the case that

$$\hat{\varphi} = X_1 \, f \tag{A.53}$$

for some vector f of length k_2, it follows that $\check{\varphi}_1 = \Xi' f$, where Ξ is defined in (A.52).

Because Ξ (and hence Ξ') is invertible, if for any $\hat{\varphi}$ of the form (A.53), $\check{\varphi}_1 = 0$, it follows that $f = 0$ and hence that $\hat{\varphi} = 0$. But it follows from (A.50) that a vector $\hat{\varphi}$ has a representation of the form (A.53) if and only if $W_2 \hat{\varphi} = 0$. Hence if any vector $\hat{\varphi}$ satisfies both $W_2 \hat{\varphi} = 0$ and $\check{\varphi}_1 = 0$, it must satisfy $\hat{\varphi} = 0$. Alternatively, if it satisfies both the linear restrictions $W_2 \hat{\varphi} = 0$ and $[I \ 0] R_2 J_2' \hat{\varphi} = 0$, it must be a zero vector. It follows from this that the matrix (A.51) must be invertible. ■

We may now proceed with the proof of Proposition 3. Let $\{\hat{\tau}_{2,t}\}$ be any process satisfying $||\hat{\tau}_2|| < \infty$. Because (3.6) is a decomposition of the form (2.20), Lemma 1 guarantees that if Assumption 2 and 4 are satisfied, then there exist non-singular matrices N_2, R_2 such that (2.29) holds. Then premultiplying (3.9) by R_2 and using (2.29) yields

$$\check{\varphi}_{1t} = \beta G E_t \check{\varphi}_{1,t+1} - \beta \check{\tau}_{1t}, \tag{A.54}$$

$$E_t \check{\varphi}_{2,t+1} = (\beta^{-1} H) \check{\varphi}_{2,t} + \check{\tau}_{2t}, \tag{A.55}$$

where

$$\check{\varphi}_t \equiv \begin{bmatrix} \check{\varphi}_{1t} \\ \check{\varphi}_{2t} \end{bmatrix} \equiv N_2^{-1} \hat{\varphi}_{2t}, \qquad \begin{bmatrix} \check{\tau}_{1t} \\ \check{\tau}_{2t} \end{bmatrix} \equiv R_2 \hat{\tau}_{2t}.$$

Because all eigenvalues of βG have modulus less than 1, (A.54) has a solution for $\{\check{\varphi}_{1t}\}$ such that $||\check{\varphi}_1|| < \infty$, given by

$$\check{\varphi}_{1t} = -\beta \sum_{j=0}^{\infty} (\beta G)^j E_t \check{\tau}_{1,t+j}. \tag{A.56}$$

Then in any period t, the value of

$$\check{\varphi}_{1t} = [I \quad 0]\, R_2 J_2' \hat{\varphi}_{2t}$$

is given by (A.56), while the value of $W_2\hat{\varphi}_{2t}$ is given by (3.12). Thus the values of both $\check{\varphi}_{1t}$ and $W_2\hat{\varphi}_{2t}$ are given as functions of variables that are exogenous and/or predetermined in period t. But Lemma 12 implies that the mapping from the linear space of vectors $\hat{\varphi}$ to the values of $\check{\varphi}_1$ and $W_2\hat{\varphi}$ is an isomorphism, so this system of equations can be uniquely solved for the value of $\hat{\varphi}_{2t}$. We thus obtain a unique solution for $\hat{\varphi}_{2t}$ as a linear function of $E_{t-1}\hat{\varphi}_{2t}$ and the $E_t\check{\tau}_{1,t+j}$ for $j \geq 0$.

This solution for $\hat{\varphi}_{2t}$ allows us to solve for $\check{\varphi}_{2t}$, and substituting this into (A.55) yields a value for $E_t\check{\varphi}_{2,t+1}$ as a linear function of $E_{t-1}\hat{\varphi}_{2t}$, $\check{\tau}_{2t}$, and the $E_t\check{\tau}_{1,t+j}$ for $j \geq 0$. The solution (A.56) implies that

$$E_t\check{\varphi}_{1,t+1} = -\beta \sum_{j=0}^{\infty} (\beta G)^j E_t \check{\tau}_{1,t+j+1}.$$

Hence we can solve for the complete vector $E_t\check{\varphi}_{t+1}$ as a linear function of $E_{t-1}\hat{\varphi}_{2t}$ and the exogenous state. Alternatively, we can solve for $E_t\hat{\varphi}_{2,t+1} = N_2 E_t\check{\varphi}_{t+1}$ as a linear function of $E_{t-1}\hat{\varphi}_{2t}$ and the exogenous state. Thus starting from an initial condition $E_{t_0-1}\hat{\varphi}_{2t_0}$, we can solve for $\hat{\varphi}_{2t_0}$ and $E_{t_0}\hat{\varphi}_{2,t_0+1}$; using this solution we can solve for $\hat{\varphi}_{2,t_0+1}$ and $E_{t_0+1}\hat{\varphi}_{2,t_0+2}$; and so on

recursively.

Thus it is possible to solve for the complete evolution $\{\hat{\varphi}_{2t}\}$ for all $t \geq t_0$, given the initial condition $E_{t_0-1}\hat{\varphi}_{2t_0}$ and the evolution of the exogenous state. By construction (A.54) and (A.55) are satisfied for each $t \geq t_0$, which implies that (3.9) is satisfied for each $t \geq t_0$. Likewise, (3.12) is satisfied for each $t > t_0$ by construction. Thus we obtain a process $\{\hat{\varphi}_{2t}\}$ that satisfies both (3.9) for all $t \geq t_0$ and (3.12) for all $t > t_0$. Moreover, because all eigenvalues of $\beta^{-1}H$ have modulus less than 1, (A.55) implies that the constructed solution satisfies $||\hat{\varphi}_2|| < \infty$.

A.8 Proof of Lemma 5

Consider the case of unidimensional policy so that $i = p = 1$, and thus $\epsilon_i = \tilde{n}$.

Suppose that (3.8) hold for all $t \geq t_0$. The first element of the vector of FOCs (3.8) can be written

$$\beta^{-1}\hat{\varphi}_{1,t}^{(1)} = \hat{\tau}_{1,t}^{(1)} \tag{A.57}$$

while the j-th element (for $2 \leq j \leq \tilde{n}$) can be written as

$$\beta^{-1}\hat{\varphi}_{1,t}^{(j)} = \hat{\tau}_{1,t}^{(j)} - E_t\hat{\varphi}_{1,t+1}^{(j-1)}. \tag{A.58}$$

This system of equations can be solved recursively for the $\{\hat{\varphi}_{1,t}^{(j)}\}$, yielding

$$\hat{\varphi}_{1,t}^{(j)} = \beta E_t \left[\Gamma^j \left(\beta L^{-1}\right) \hat{\tau}_{1,t}\right] \tag{A.59}$$

for each $1 \leq j \leq \tilde{n}$, where $\Gamma^j(\mu)$ is the j-th row of the matrix polynomial $\Gamma(\mu)$. This gives the vector of conditions (3.16).

The $(\tilde{n}+1)$-st element of the vector of FOCs (3.8) states that

$$E_t \hat{\varphi}_{1,t+1}^{(\tilde{n})} = \hat{\tau}_{1,t}^{(\tilde{n}+1)}. \tag{A.60}$$

Substituting solution (A.59) for $\hat{\varphi}_{1,t}^{(j)}$, we obtain

$$\hat{\tau}_{1,t}^{(\tilde{n}+1)} = E_t \left[\beta L^{-1} \Gamma^{\tilde{n}} \left(\beta L^{-1}\right) \hat{\tau}_{1,t}\right] \tag{A.61}$$

which implies (3.17). Here we use the fact that

$$\delta(\mu)' = (-\mu)^{\tilde{n}} \left[e'_{\tilde{n}+1} - \mu^{-1} \Gamma^{\tilde{n}} \left(\mu^{-1}\right)\right]$$

where $e'_{\tilde{n}+1}$ is a $1 \times (\tilde{n}+1)$ vector of the form $e'_{\tilde{n}+1} \equiv [0...0 \;\; 1]$. Thus both (3.16) and (3.17) must hold in all periods $t \geq t_0$.

To prove the converse, suppose that the processes $\{\hat{\tau}_{1,t}\}$ and $\{\hat{\varphi}_{1,t}\}$ satisfy (3.16) and (3.17) in all periods $t \geq t_0$. Condition (3.16) implies (A.59) for each $1 \leq j \leq \tilde{n}$, which in turn implies condition (A.57), and condition (A.58) for each $2 \leq j \leq \tilde{n}$. These are just the first $\tilde{n}$ elements of the vector of FOCs (3.8). Condition (3.17) implies (A.61), which together with the case $j = \tilde{n}$ of (A.59) implies (A.60). This is just the $(\tilde{n}+1)$-st element of the vector of FOCs (3.8). Thus the entire vector of conditions (3.8) must hold in each period $t \geq t_0$.

A.9 Proof of Proposition 4

To prove Proposition 4, it will be useful to appeal to the following Lemma.

Lemma 13 *Suppose that the processes* $\{\hat{\tau}_{1,t}\}$ *and* $\{\hat{\varphi}_{1,t}\}$ *satisfy (3.16) for all* $t \geq t_0$. *Then for any* $1 \leq j \leq \tilde{n}$, *and any* $t \geq t_0$,

$$E_t z_{t+j-1} - E_{t-1} z_{t+j-1} = -(-\beta)^{-j} \left[\hat{\varphi}_{1,t}^{(j)} - E_{t-1} \hat{\varphi}_{1,t}^{(j)}\right], \tag{A.62}$$

where $\hat{\varphi}_{1,t}^{(j)}$ *is the* j*-th element of the vector* $\hat{\varphi}_{1,t}$. *When* $t = t_0$, *the expression* $E_{t_0-1} \hat{\varphi}_{1,t_0}^{(j)}$ *is taken to refer to the historical expectations* $\beta E_{t_0-1}[\Gamma^j(\beta L^{-1}) \hat{\tau}_{1,t_0}]$.

Proof. Using the definition (3.15) we have

$$\begin{aligned}
E_t z_{t+j-1} - E_{t-1} z_{t+j-1} &= E_t \left[\hat{\tau}_{1,t+j-1}^{(1)} - \beta^{-1} \hat{\tau}_{1,t+j-2}^{(2)} + ... + \left(-\beta^{-1}\right)^{j-1} \hat{\tau}_{1,t}^{(j)}\right] \\
&\quad - E_{t-1} \left[\hat{\tau}_{1,t+j-1}^{(1)} - \beta^{-1} \hat{\tau}_{1,t+j-2}^{(2)} + ... + \left(-\beta^{-1}\right)^{j-1} \hat{\tau}_{1,t}^{(j)}\right] \\
&= \left(-\beta^{-1}\right)^{j-1} \left\{E_t \left[\Gamma^j \left(\beta L^{-1}\right) \hat{\tau}_{1,t}\right] - E_{t-1} \left[\Gamma^j \left(\beta L^{-1}\right) \hat{\tau}_{1,t}\right]\right\} \\
&= -(-\beta)^{-j} \left[\hat{\varphi}_{1,t}^{(j)} - E_{t-1} \hat{\varphi}_{1,t}^{(j)}\right].
\end{aligned}$$

Here, the first equality uses the definition of $\delta(\mu)'$, and the fact that $E_t\hat{\tau}_{1,t-k} = E_{t-1}\hat{\tau}_{1,t-k}$ for any $k \geq 1$. The second equality uses the definition of $\Gamma(\mu)$, denotes by $\Gamma^j(\mu)$ its j-th row. The third equality uses (3.16). In the case that $t = t_0$, the replacement of $\beta E_{t-1}\left[\Gamma^j\left(\beta L^{-1}\right)\hat{\tau}_{1,t}\right]$ by $E_{t-1}\hat{\varphi}_{1,t}^{(j)}$ is justified under the definition of $E_{t_0-1}\hat{\varphi}_{1,t_0}^{(j)}$ proposed above. ■

We may now proceed with the proof of Proposition 4. Condition (3.8) implies that (3.16) and (3.17) must hold for all $t \geq t_0$, using Lemma 5. The fact that (3.16) holds implies that (A.62) must also hold for all $t \geq t_0$, using Lemma 13.

Let us first consider any period $t \geq t_0 + \tilde{n} - k_1$. Then

$$\begin{aligned} E_t z_{t+k_1} &= \sum_{j=1}^{\tilde{n}-k_1}\left(E_{t+1-j}z_{t+k_1} - E_{t-j}z_{t+k_1}\right) + E_{t-(\tilde{n}-k_1)}z_{t+k_1} \\ &= \sum_{j=1}^{\tilde{n}-k_1}(-\beta)^{-(k_1+j)}\left[\hat{\varphi}_{1,t+1-j}^{(k_1+j)} - E_{t-j}\hat{\varphi}_{1,t+1-j}^{(k_1+j)}\right]. \end{aligned} \tag{A.63}$$

Here the second line uses (A.62) to replace the first term on the right-hand side of the first line, and uses (3.17) to eliminate the second term.

Given Assumption 6, (3.11) implies that the entire vector $\left[\hat{\varphi}_{1,t} - E_{t-1}\hat{\varphi}_{1,t}\right]$ can be reconstructed from its first k_1 elements, using

$$\hat{\varphi}_{1,t} - E_{t-1}\hat{\varphi}_{1,t} = \begin{bmatrix} I_{k_1} \\ \Phi \end{bmatrix}\left[\bar{\varphi}_{1,t} - E_{t-1}\bar{\varphi}_{1,t}\right] \tag{A.64}$$

for any $t \geq t_0 + 1$, where $\bar{\varphi}_{1,t}$ is the vector consisting of the first k_1 elements of $\hat{\varphi}_{1,t}$. Using (A.64) to substitute for the terms on the right-hand side of (A.63), we obtain

$$\begin{aligned} E_t z_{t+k_1} &= \sum_{j=1}^{\tilde{n}-k_1} -(-\beta)^{-(k_1+j)}\,\phi_j'\left[\bar{\varphi}_{1,t+1-j} - E_{t-j}\bar{\varphi}_{1,t+1-j}\right] \\ &= \sum_{j=1}^{\tilde{n}-k_1}(-\beta)^{-(k_1+j)}\,\phi_j' w_{t+1-j} \\ &= (-\beta)^{-k_1}\sum_{j=1}^{\tilde{n}-k_1}\phi_j'\Omega_t^j \\ &= (-\beta)^{-k_1}\,tr\left[\Phi\Omega_t\right], \end{aligned} \tag{A.65}$$

which establishes (3.20). Here we use the notation ϕ_j' for the j-th row of Φ and the notation Ω_t^j for the j-th column of Ω_t. In addition, the second line uses Lemma 13 to substitute for the elements of $\bar{\varphi}_{1,t+1-j} - E_{t-j}\bar{\varphi}_{1,t+1-j}$, and definition (3.18), while the third line uses the definition of Ω_t.

Let us now consider any period $t_0 \leq t < t_0 + \tilde{n} - k_1$. Then

$$\begin{aligned} E_t z_{t+k_1} &= \sum_{j=1}^{t-t_0} \left(E_{t+1-j} z_{t+k_1} - E_{t-j} z_{t+k_1}\right) + E_{t_0} z_{t+k_1} \\ &= (-\beta)^{-k_1} \sum_{j=1}^{t-t_0} \phi_j' \Omega_t^j + E_{t_0} z_{t+k_1}, \end{aligned} \tag{A.66}$$

where the second line is obtained using the same reasoning as was used to derive (A.63) and (A.65).

For the given historical expectations e_{t_0-1}, let Ξ_{1,t_0-1} be given by (3.21) where χ_{t_0-1} is the vector whose j-th element is given by (3.22). With this definition, (3.13) together with (3.8) and Lemma 5 imply that

$$V_2 \left[\hat{\varphi}_{1,t_0}\right] = V_2 E_{t_0-1} \left[\hat{\varphi}_{1,t_0}\right] + V_{22} \chi_{t_0-1}.$$

Premultiplying by V_{22}^{-1} and noting that $\Phi \equiv -V_{22}^{-1} V_{21}$, this yields

$$\left[-\Phi \quad I_{\tilde{n}-k_1}\right] \left[\hat{\varphi}_{1,t_0} - E_{t_0-1} \hat{\varphi}_{1,t_0}\right] = \chi_{t_0-1},$$

which can alternatively be written

$$\left[\Phi \quad (-I_{\tilde{n}-k_1})\right] w_{t_0} = \chi_{t_0-1}, \tag{A.67}$$

using Lemma 13 and definition (3.18).

For any $1 \leq j \leq \tilde{n} - k_1$, the j-th row of (A.67) can be written

$$\begin{aligned} &\phi_j' \bar{w}_{t_0} - (-\beta)^{k_1+j} \left[E_{t_0} z_{t_0+k_1+j-1} - E_{t_0-1} z_{t_0+k_1+j-1}\right] \\ = \ &(-\beta)^{k_1+j} E_{t_0-1} z_{t_0+k_1+j-1} - \sum_{i=1}^{\tilde{n}-k_1-j} (-\beta)^{-i} \phi_{j+i}' \bar{w}_{t_0-i}, \end{aligned}$$

or alternatively,

$$(-\beta)^{k_1+j} E_{t_0} z_{t_0+k_1+j-1} = \sum_{i=0}^{\tilde{n}-k_1-j} (-\beta)^{-i} \phi_{j+i}' \bar{w}_{t_0-i} = \sum_{h=j}^{\tilde{n}-k_1} (-\beta)^{j} \phi_h' \Omega_{t_0+j-1}^h.$$

Thus if we let $j = t + 1 - t_0$, we find that

$$E_{t_0} z_{t+k_1} = (-\beta)^{-k_1} \sum_{h=t-t_0+1}^{\tilde{n}-k_1} \phi_h' \Omega_t^h.$$

Using this to substitute for the final term on the right-hand side of (A.66), we obtain

$$E_t z_{t+k_1} = (-\beta)^{-k_1} \sum_{j=1}^{\tilde{n}-k_1} \phi_j' \Omega_t^j,$$

so that (3.20) is satisfied for each $t_0 \leq t < t_0 + \tilde{n} - k_1$. Since we have already shown that (3.20) holds for any $t \geq t_0 + \tilde{n} - k_1$, it follows that (3.20) is satisfied for each $t \geq t_0$.

A.10 Proof of Lemma 6

For period t_0, conditions (3.23) and (3.24) hold by assumption, given the initial condition (3.13) and the initial Lagrange multipliers (3.21). Next, (3.8) implies $\hat{\varphi}_{1,t} = \beta E_t[\Gamma(\beta L^{-1})\hat{\tau}_{1,t}]$ at all dates $t \geq t_0$, by Lemma 5. Using this, conditions (3.11) in turn imply that

$$V_2\,\hat{\varphi}_{1,t} = V_2\,E_{t-1}\hat{\varphi}_{1,t} = V_2\beta E_{t-1}[\Gamma(\beta L^{-1})\hat{\tau}_{1,t}]$$

for all $t > t_0$. Equation (3.23) thus holds in all period $t \geq t_0$, where $\Xi_{1,t-1}$ is given by (3.24) in period $t = t_0$, and

$$\Xi_{1,t-1} = V_2\beta E_{t-1}[\Gamma(\beta L^{-1})\hat{\tau}_{1,t}]$$

for all $t > t_0$. To prove that (3.24) holds at all dates, we need to show that $V_{22}\chi_{t-1} = 0$ for all periods $t > t_0$.

Given the initial Lagrange multipliers Ξ_{1,t_0} defined in (3.21) and using Proposition 4 implies that (3.20) must hold at all dates $t \geq t_0$. Condition (3.20) then implies that for any $t \geq t_0$ and any $1 \leq j \leq \tilde{n} - k_1$,

$$\begin{aligned}
E_t z_{t+k_1+j-1} &= E_t\left[E_{t+j-1} z_{t+k_1+j-1}\right] = (-\beta)^{-k_1}\,\mathrm{tr}[\Phi\,E_t\Omega_{t+j-1}] \\
&= (-\beta)^{-k_1}\,E_t\left[(-\beta)^{-1}\,\phi_1'\bar{w}_{t+j-1} + \ldots + (-\beta)^{-(\tilde{n}-k_1)}\,\phi_{\tilde{n}-k_1}'\bar{w}_{t+j+k_1-\tilde{n}}\right] \\
&= (-\beta)^{-k_1}\left[(-\beta)^{-j}\,\phi_j'\bar{w}_t + \ldots + (-\beta)^{-(\tilde{n}-k_1)}\,\phi_{\tilde{n}-k_1}'\bar{w}_{t+j+k_1-\tilde{n}}\right] \\
&= (-\beta)^{-(k_1+j)}\sum_{i=0}^{\tilde{n}-k_1-j}(-\beta)^{-i}\,\phi_{j+i}'\bar{w}_{t-i}. \qquad \text{(A.68)}
\end{aligned}$$

Using this to substitute for $E_{t-1}z_{t+k_1+j-1}$ in (3.22), we obtain

$$\chi_{t-1}^j = \phi_j' E_{t-1}\bar{w}_t = 0$$

for all $t > t_0$, and any $1 \leq j \leq \tilde{n} - k_1$. It follows that $\chi_{t-1} = 0$, and hence that $V_{22}\chi_{t-1} = 0$ for all $t > t_0$.

A.11 Proof of Proposition 5

Let the process $\{\hat{\varphi}_{1,t}\}$ be given by (3.16) for all $t \geq t_0$. Note that if $k_1 < \tilde{n}$, satisfaction of (3.20) for all $t \geq t_0$ implies that

$$E_t z_{t+\tilde{n}} = E_t[E_{t+\tilde{n}-k_1} z_{t+\tilde{n}}] = (-\beta)^{-k_1}\mathrm{tr}[\Phi\,E_t\Omega_{t+\tilde{n}-k_1}] = 0$$

for any $t \geq t_0$, using the fact that $E_t\bar{w}_{t+j} = 0$ for any $t \geq t_0$ and any $j \geq 1$. But if $k_1 = \tilde{n}$, Ω_t is a null matrix, and (3.20) also implies that $E_t z_{t+\tilde{n}} = 0$ in this case as well. Hence the target criterion (3.20) implies condition (3.17). Then by Lemma 5, the fact that (3.16) and (3.17) hold for all $t \geq t_0$ implies that (3.8) holds for all $t \geq t_0$.

Recall from (A.68) that condition (3.20) also implies

$$E_t z_{t+k_1+j-1} = (-\beta)^{-(k_1+j)}\sum_{i=0}^{\tilde{n}-k_1-j}(-\beta)^{-i}\,\phi_{j+i}'\bar{w}_{t-i}$$

for any $t \geq t_0$ and any $1 \leq j \leq \tilde{n} - k_1$. It follows that for all $t > t_0$ and all $1 \leq j \leq \tilde{n} - k_1$:

$$E_t z_{t+k_1+j-1} - E_{t-1} z_{t+k_1+j-1} = (-\beta)^{-(k_1+j)} \phi_j' \bar{w}_t$$

so that

$$w_t \equiv \begin{bmatrix} (-\beta)\left[z_t - E_{t-1} z_t\right] \\ \vdots \\ (-\beta)^{k_1}[E_t z_{t+k_1-1} - E_{t-1} z_{t+k_1-1}] \\ (-\beta)^{k_1+1}[E_t z_{t+k_1} - E_{t-1} z_{t+k_1}] \\ \vdots \\ (-\beta)^{\tilde{n}}[E_t z_{t+\tilde{n}-1} - E_{t-1} z_{t+\tilde{n}-1}] \end{bmatrix} = \begin{bmatrix} I_{k_1} \\ \Phi \end{bmatrix} \bar{w}_t.$$

Premultiplying by V_2 implies in turn that

$$V_2 w_t = [V_{21} \quad V_{22}] \begin{bmatrix} I_{k_1} \\ \Phi \end{bmatrix} \bar{w}_t = 0$$

for any $t > t_0$. For the process $\{\hat{\varphi}_{1,t}\}$ given by (3.16) for all $t \geq t_0$, we can apply Lemma 13 to express w_t as

$$w_t = -\left[\hat{\varphi}_{1,t} - E_{t-1}\hat{\varphi}_{1,t}\right]$$

for all $t \geq t_0$. Since $V_2 w_t = 0$ for any $t > t_0$, equation (3.11) must hold for all $t > t_0$.

To show that the initial condition (3.13) also holds, where the vector Ξ_{1,t_0-1} is defined by (3.21), note that (A.68) implies

$$(-\beta)^{k_1+j} E_{t_0} z_{t_0+k_1+j-1} = \sum_{i=0}^{\tilde{n}-k_1-j} (-\beta)^{-i} \phi_{j+i}' \bar{w}_{t_0-i}$$

for any $1 \leq j \leq \tilde{n} - k_1$. Subtracting $\phi_j' \bar{w}_{t_0} + (-\beta)^{k_1+j} E_{t_0-1} z_{t_0+k_1+j-1}$ on both sides (where again expectations taken at date $t_0 - 1$ denote historical forecasts) yields

$$-\phi_j' \bar{w}_{t_0} + (-\beta)^{k_1+j} \left[E_{t_0} z_{t_0+k_1+j-1} - E_{t_0-1} z_{t_0+k_1+j-1}\right] = -\chi_{t_0-1}^j$$

where $\chi_{t_0-1}^j$ is the j-th element of the vector χ_{t_0-1}, defined in (3.22). Since the previous expression holds for any $1 \leq j \leq \tilde{n} - k_1$, we may rewrite it in matrix form as

$$[-\Phi \quad I_{\tilde{n}-k_1}]\, w_{t_0} = -\chi_{t_0-1},$$

using definition (3.18). Using again definition (3.18) and Lemma 13 we can equivalently rewrite this as

$$[-\Phi \quad I_{\tilde{n}-k_1}] \left[\hat{\varphi}_{1,t_0} - E_{t_0-1}\hat{\varphi}_{1,t_0}\right] = \chi_{t_0-1},$$

or as

$$V_2 \hat{\varphi}_{1,t_0} = V_2 E_{t_0-1}\left[\hat{\varphi}_{1,t_0}\right] + V_{22}\chi_{t_0-1},$$

after premultiplying on both sides by V_{22}. Using (3.16) to replace $\hat{\varphi}_{1,t_0}$ with $\beta E_{t_0}\left[\Gamma(\beta L^{-1})\hat{\tau}_{1,t_0}\right]$ on the right-hand side yields the initial condition (3.13), where the vector Ξ_{1,t_0-1} is defined by (3.21).

B Linearization of the FOCs

In this section, we provide a linearization of the first-order conditions (1.8).

B.1 Linearizing $(D_1\pi(y_t,\xi_t))'$.

We note that $(D_1\pi(y_t,\xi_t))'$ is a $(m\times 1)$ vector-valued function. A first-order approximation of this vector function yields

$$(D_1\pi(y_t,\xi_t))' = (D_1\pi)' + D_1\left[(D_1\pi)'\right](y_t-\bar{y}) + D_2\left[(D_1\pi)'\right]\left(\xi_t-\bar{\xi}\right) + \mathcal{O}(\epsilon^2).$$

B.2 Linearizing $(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t$.

To approximate the vector functions $(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t$, recall that the matrix function $(D_1F(y_t,\xi_t;y_{t-1}))'$ is $(m\times k)$ and θ_t is a $(k\times 1)$ vector. It will be convenient to rewrite $(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t$ as follows

$$\begin{aligned}(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t &= vec\left[(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t\right] = vec\left[I_m(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t\right]\\ &= (\theta_t'\otimes I_m)\quad vec\left[(D_1F(y_t,\xi_t;y_{t-1}))'\right].\end{aligned}$$

We can then write the $(m\times km)$ matrix $(\theta_t'\otimes I_m)$ as

$$\begin{aligned}(\theta_t'\otimes I_m) &= \left(\bar{\theta}'\otimes I_m\right) + (\theta_t'\otimes I_m) - \left(\bar{\theta}'\otimes I_m\right)\\ &= \left(\bar{\theta}'\otimes I_m\right) + \left(\tilde{\theta}_t'\otimes I_m\right).\end{aligned}$$

where $\tilde{\theta}_t=\theta_t-\bar{\theta}$. Next, the $(km\times 1)$ function $vec\left[(D_1F(y_t,\xi_t;y_{t-1}))'\right]$ can be approximated as follows

$$\begin{aligned}vec\left[(D_1F(y_t,\xi_t;y_{t-1}))'\right] &= vec\left[(D_1F)'\right] + \frac{\partial vec\left[(D_1F)'\right]}{\partial y_t'}(y_t-\bar{y}) + \frac{\partial vec\left[(D_1F)'\right]}{\partial \xi_t'}\left(\xi_t-\bar{\xi}\right)\\ &\quad + \frac{\partial vec\left[(D_1F)'\right]}{\partial y_t'}(y_{t-1}-\bar{y}) + \mathcal{O}(\epsilon^2)\end{aligned}$$

which can be written as

$$vec\left[(D_1F(y_t,\xi_t;y_{t-1}))'\right] = vec\left[(D_1F)'\right] + D_1\left[(D_1F)'\right]\tilde{y}_t + D_3\left[(D_1F)'\right]\tilde{y}_{t-1} + D_2\left[(D_1F)'\right]\tilde{\xi}_t + \mathcal{O}(\epsilon^2).$$

A first-order approximation of $(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t$ is thus given by

$$\begin{aligned}&(D_1F(y_t,\xi_t;y_{t-1}))'\theta_t\\ &= \left[\left(\bar{\theta}'\otimes I_m\right) + \left(\tilde{\theta}_t'\otimes I_m\right)\right]\times\Big\{vec\left[(D_1F)'\right] + D_1\left[(D_1F)'\right]\tilde{y}_t\\ &+ D_3\left[(D_1F)'\right]\tilde{y}_{t-1} + D_2\left[(D_1F)'\right]\tilde{\xi}_t + \mathcal{O}(\epsilon^2)\Big\}\\ &= (\bar{\theta}'\otimes I_m)\{vec\left[(D_1F)'\right] + D_1\left[(D_1F)'\right]\tilde{y}_t + D_3\left[(D_1F)'\right]\tilde{y}_{t-1} + D_2\left[(D_1F)'\right]\tilde{\xi}_t\}\\ &+ (\tilde{\theta}_t'\otimes I_m)\{vec\left[(D_1F)'\right] + D_1\left[(D_1F)'\right]\tilde{y}_t + D_3\left[(D_1F)'\right]\tilde{y}_{t-1} + D_2\left[(D_1F)'\right]\tilde{\xi}_t\}\\ &+ \mathcal{O}(\epsilon^2)\end{aligned}$$

Note that several elements of the last two lines are of order $\mathcal{O}(\epsilon^2)$. They can thus be omitted as part of the approximation error. Furthermore, as $\left(\bar{\theta}' \otimes I_m\right) vec\left[(D_1F)'\right] = vec\left[I_m (D_1F)' \bar{\theta}\right]$ $= (D_1F)' \bar{\theta}$, we can rewrite

$$\begin{aligned}
(D_1F(y_t,\xi_t;y_{t-1}))' \theta_t &= \left(\bar{\theta}' \otimes I_m\right)\{D_1\left[(D_1F)'\right]\tilde{y}_t + D_3\left[(D_1F)'\right]\tilde{y}_{t-1} + D_2\left[(D_1F)'\right]\tilde{\xi}_t\} \\
&\quad + (D_1F)'\bar{\theta} + (D_1F)'\tilde{\theta}_t + \mathcal{O}(\epsilon^2).
\end{aligned}$$

B.3 Linearization of the FOCs

Having linearized the two vectors described above, we are now prepared to provide a first-order approximation to (1.7):

$$\begin{aligned}
0 &= D_1\left[(D_1\pi)'\right]\tilde{y}_t + D_2\left[(D_1\pi)'\right]\tilde{\xi}_t \\
&\quad + (D_1F)'\tilde{\theta}_t + \left(\bar{\theta}' \otimes I_m\right)\left\{D_1\left[(D_1F)'\right]\tilde{y}_t + D_3\left[(D_1F)'\right]\tilde{y}_{t-1} + D_2\left[(D_1F)'\right]\tilde{\xi}_t\right\} \\
&\quad + \beta E_t\left\{(D_3F)'\tilde{\theta}_{t+1} + \left(\bar{\theta}' \otimes I_m\right)\left\{D_1\left[(D_3F)'\right]\tilde{y}_{t+1} + D_3\left[(D_3F)'\right]\tilde{y}_t + D_2\left[(D_3F)'\right]\tilde{\xi}_{t+1}\right\}\right\} \\
&\quad + (D_1g)'\tilde{\Theta}_t + \left(\bar{\Theta}' \otimes I_m\right)\left\{D_1\left[(D_1g)'\right]\tilde{y}_t + D_3\left[(D_1g)'\right]E_t\tilde{y}_{t+1} + D_2\left[(D_1g)'\right]\tilde{\xi}_t\right\} \\
&\quad + \beta^{-1}(D_3g)'\tilde{\Theta}_{t-1} + \left(\bar{\Theta}' \otimes I_m\right)\beta^{-1}\left\{D_1\left[(D_3g)'\right]\tilde{y}_{t-1} + D_3\left[(D_3g)'\right]\tilde{y}_t + D_2\left[(D_3g)'\right]\tilde{\xi}_{t-1}\right\} \\
&\quad + (D_1\pi)' + (D_1F)'\bar{\theta} + \beta(D_3F)'\bar{\theta} + (D_1g)'\bar{\Theta} + \beta^{-1}(D_3g)'\bar{\Theta} + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Using the fact that in steady state

$$0 = (D_1\pi)' + (D_1F)'\bar{\theta} + \beta(D_3F)'\bar{\theta} + (D_1g)'\bar{\Theta} + \beta^{-1}(D_3g)'\bar{\Theta},$$

the last line reduces to the approximation error.

Let us define

$$\begin{aligned}
S &\equiv D_1\left[(D_1\pi)'\right] + \left(\bar{\theta}' \otimes I_m\right)\{D_1\left[(D_1F)'\right] + \beta D_3\left[(D_3F)'\right]\} \\
&\quad + \left(\bar{\Theta}' \otimes I_m\right)\{D_1\left[(D_1g)'\right] + \beta^{-1}D_3\left[(D_3g)'\right]\} \\
R &\equiv \left(\bar{\theta}' \otimes I_m\right)D_3\left[(D_1F)'\right] + \left(\bar{\Theta}' \otimes I_m\right)\beta^{-1}D_1\left[(D_3g)'\right] \\
R^\dagger &\equiv \left(\bar{\theta}' \otimes I_m\right)D_1\left[(D_3F)'\right] + \left(\bar{\Theta}' \otimes I_m\right)\beta^{-1}D_3\left[(D_1g)'\right]
\end{aligned}$$

and the matrix polynomial

$$\begin{aligned}
B(L) &\equiv \left\{D_2\left[(D_1\pi)'\right] + \left(\bar{\theta}' \otimes I_m\right)D_2\left[(D_1F)'\right] + \left(\bar{\Theta}' \otimes I_m\right)D_2\left[(D_1g)'\right]\right\}\cdot L \\
&\quad + \beta^{-1}\left(\bar{\Theta}' \otimes I_m\right)D_2\left[(D_3g)'\right]\cdot L^2 + \beta\left(\bar{\theta}' \otimes I_m\right)D_2\left[(D_3F)'\right].
\end{aligned}$$

Using this, we can rewrite the approximation to the first-order conditions as:

$$\begin{aligned}
0 &= S\tilde{y}_t + R\tilde{y}_{t-1} + \beta R^\dagger E_t\tilde{y}_{t+1} + E_t\left[B(L)\tilde{\xi}_{t+1}\right] \\
&\quad + (D_3F)'\beta E_t\tilde{\theta}_{t+1} + (D_1g)'\tilde{\Theta}_t + (D_1F)'\tilde{\theta}_t + \beta^{-1}(D_3g)'\tilde{\Theta}_{t-1} + \mathcal{O}(\epsilon^2) \\
&= S\tilde{y}_t + R\tilde{y}_{t-1} + \beta R^\dagger E_t\tilde{y}_{t+1} + E_t\left[B(L)\tilde{\xi}_{t+1}\right] + E_t\left[\left(\bar{A} - \beta^{-1}\bar{I}L\right)'\tilde{\varphi}_{t+1}\right] + \mathcal{O}(\epsilon^2).
\end{aligned}$$

These conditions further reduce to (2.16), using the following lemma.

Lemma 14 *If $\pi()$, $F()$ and $g()$ are twice continuously differentiable, then S is a symmetric matrix (i.e., $S = S'$) and $R^{\dagger} = R'$.*

Proof. We start by showing that

$$\left(\bar{\theta}' \otimes I_m\right) D_1 \left[(D_3 F)'\right] = \left\{\left(\bar{\theta}' \otimes I_m\right) D_3 \left[(D_1 F)'\right]\right\}'.$$

Expanding the derivatives, we have

$$D_1 \left[(D_3 F)'\right] = \frac{\partial vec\left[\left(\frac{\partial F}{\partial y'_{t-1}}\right)'\right]}{\partial y'_t} = \frac{\partial vec\begin{bmatrix} \partial F_1/\partial y_{1,t-1} & \cdots & \partial F_k/\partial y_{1,t-1} \\ \vdots & & \vdots \\ \partial F_1/\partial y_{m,t-1} & \cdots & \partial F_k/\partial y_{m,t-1} \end{bmatrix}}{\partial y'_t}$$

so that

$$D_1 \left[(D_3 F)'\right] = \begin{bmatrix} \frac{\partial^2 F_1}{\partial y_{1,t-1}\partial y_{1,t}} & \cdots & \frac{\partial^2 F_1}{\partial y_{1,t-1}\partial y_{m,t}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_1}{\partial y_{m,t-1}\partial y_{1,t}} & \cdots & \frac{\partial^2 F_1}{\partial y_{m,t-1}\partial y_{m,t}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_k}{\partial y_{1,t-1}\partial y_{1,t}} & \cdots & \frac{\partial^2 F_k}{\partial y_{1,t-1}\partial y_{m,t}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_k}{\partial y_{m,t-1}\partial y_{1,t}} & \cdots & \frac{\partial^2 F_k}{\partial y_{m,t-1}\partial y_{m,t}} \end{bmatrix}.$$

By Young's theorem, if $F()$ is twice continuously differentiable, then $\partial^2 F_j / (\partial y_{k,t-1}\partial y_{l,t}) = \partial^2 F_j / (\partial y_{l,t}\partial y_{k,t-1})$ for all j, k, l. It follows that

$$\begin{aligned}
&\left(\bar{\theta}' \otimes I_m\right) D_1 \left[(D_3 F)'\right] \\
&= \begin{bmatrix} \bar{\theta}_1 & & 0 & & \bar{\theta}_m & & 0 \\ & \ddots & & \cdots & & \ddots & \\ 0 & & \bar{\theta}_1 & & 0 & & \bar{\theta}_m \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{1,t-1}} & \cdots & \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{1,t-1}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{m,t-1}} & \cdots & \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{m,t-1}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{1,t-1}} & \cdots & \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{1,t-1}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{m,t-1}} & \cdots & \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{m,t-1}} \end{bmatrix} \\
&= \begin{bmatrix} \bar{\theta}_1 \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{1,t-1}} + \cdots + \bar{\theta}_m \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{1,t-1}} & \cdots & \bar{\theta}_1 \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{1,t-1}} + \cdots + \bar{\theta}_m \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{1,t-1}} \\ \bar{\theta}_1 \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{2,t-1}} + \cdots + \bar{\theta}_m \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{2,t-1}} & & \bar{\theta}_1 \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{2,t-1}} + \cdots + \bar{\theta}_m \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{2,t-1}} \\ \vdots & & \vdots \\ \bar{\theta}_1 \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{m,t-1}} + \cdots + \bar{\theta}_m \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{m,t-1}} & \cdots & \bar{\theta}_1 \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{m,t-1}} + \cdots + \bar{\theta}_m \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{m,t-1}} \end{bmatrix}.
\end{aligned}$$

On the other hand:

$$D_3\left[(D_1F)'\right] = \frac{\partial vec\begin{bmatrix} \partial F_1/\partial y_{1,t} & \cdots & \partial F_k/\partial y_{1,t} \\ \vdots & & \vdots \\ \partial F_1/\partial y_{m,t} & \cdots & \partial F_k/\partial y_{m,t}\end{bmatrix}}{\partial y'_{t-1}}$$

so that using again Young's theorem, we have

$$\left\{\left(\bar{\theta}'\otimes I_m\right)D_3\left[(D_1F)'\right]\right\}'$$

$$= \left\{\begin{bmatrix} \bar{\theta}_1 & 0 & & \bar{\theta}_m & 0 \\ & \ddots & \cdots & & \ddots & \\ 0 & \bar{\theta}_1 & & 0 & \bar{\theta}_m \end{bmatrix}\begin{bmatrix} \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{1,t-1}} & \cdots & \frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{m,t-1}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{1,t-1}} & \cdots & \frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{m,t-1}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{1,t-1}} & \cdots & \frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{m,t-1}} \\ \vdots & & \vdots \\ \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{1,t-1}} & \cdots & \frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{m,t-1}} \end{bmatrix}\right\}'$$

$$= \begin{bmatrix} \bar{\theta}_1\frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{1,t-1}} + \cdots + \bar{\theta}_m\frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{1,t-1}} & \cdots & \bar{\theta}_1\frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{1,t-1}} + \cdots + \bar{\theta}_m\frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{1,t-1}} \\ \vdots & & \vdots \\ \bar{\theta}_1\frac{\partial^2 F_1}{\partial y_{1,t}\partial y_{m,t-1}} + \cdots + \bar{\theta}_m\frac{\partial^2 F_k}{\partial y_{1,t}\partial y_{m,t-1}} & \cdots & \bar{\theta}_1\frac{\partial^2 F_1}{\partial y_{m,t}\partial y_{m,t-1}} + \cdots + \bar{\theta}_m\frac{\partial^2 F_k}{\partial y_{m,t}\partial y_{m,t-1}} \end{bmatrix}$$

which is equal to the expression derived before for $\left(\bar{\theta}'\otimes I_m\right)D_1\left[(D_3F)'\right]$.

Having shown that $\left(\bar{\theta}'\otimes I_m\right)D_1\left[(D_3F)'\right] = \left\{\left(\bar{\theta}'\otimes I_m\right)D_3\left[(D_1F)'\right]\right\}'$, it is then easy to see (when replacing $\bar{\theta}$ with $\bar{\Theta}$ and F with g) that

$$\left(\bar{\Theta}'\otimes I_m\right)D_1\left[(D_3g)'\right] = \left\{\left(\bar{\Theta}'\otimes I_m\right)D_3\left[(D_1g)'\right]\right\}'.$$

It follows that $R^\dagger = R'$.

A similar derivation can be performed to show that

$$\begin{aligned}\left(\bar{\theta}'\otimes I_m\right)D_j\left[(D_jF)'\right] &= \left\{\left(\bar{\theta}'\otimes I_m\right)D_j\left[(D_jF)'\right]\right\}' \\ \left(\bar{\Theta}'\otimes I_m\right)D_j\left[(D_jg)'\right] &= \left\{\left(\bar{\Theta}'\otimes I_m\right)D_j\left[(D_jg)'\right]\right\}'\end{aligned}$$

for $j = 1, 3$, and $D_1\left[(D_1\pi)'\right] = \left\{D_1\left[(D_1\pi)'\right]\right\}'$. This implies that the matrix S is symmetric. ■

C Second-Order Conditions

We now perform a second-order approximation of the Lagrangian (1.6) around the optimal steady state. This involves a second-order approximation of each of the three terms in the square brackets of (1.6), evaluating the derivatives at the optimal steady state.

C.1 Second-Order Approximation of the Lagrangian

A second-order approximation of the function $\pi(y_t,\xi_t)$ yields

$$\begin{aligned}\pi(y_t,\xi_t) &= \pi\left(\bar{y},\bar{\xi}\right)+\begin{bmatrix} D_1\pi & D_2\pi\end{bmatrix}\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\end{bmatrix}\\ &\quad+\frac{1}{2}\begin{bmatrix}\tilde{y}_t' & \tilde{\xi}_t'\end{bmatrix}\begin{bmatrix} D_1\left[(D_1\pi)'\right] & D_2\left[(D_1\pi)'\right]\\ D_1\left[(D_2\pi)'\right] & D_2\left[(D_2\pi)'\right]\end{bmatrix}\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\end{bmatrix}+\mathcal{O}(\epsilon^3).\end{aligned}$$

A second-order approximation of the function $\theta_t'F(y_t,\xi_t;y_{t-1})$ yields

$$\begin{aligned}\theta_t'F(y_t,\xi_t;y_{t-1}) &= \bar{\theta}'\begin{bmatrix} D_1F & D_2F & D_3F\end{bmatrix}\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\\ \tilde{y}_{t-1}\end{bmatrix}\\ &\quad+\frac{1}{2}\begin{bmatrix}\tilde{y}_t' & \tilde{\xi}_t' & \tilde{y}_{t-1}' & \tilde{\theta}_t'\end{bmatrix}\mathcal{H}\left(\bar{\theta}'F\right)\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\\ \tilde{y}_{t-1}\\ \tilde{\theta}_t\end{bmatrix}+\mathcal{O}(\epsilon^3)\end{aligned}$$

where the Hessian matrix

$$\mathcal{H}\left(\bar{\theta}'F\right)=\begin{bmatrix}\left(\bar{\theta}'\otimes I_m\right)D_1\left[(D_1F)'\right] & \left(\bar{\theta}'\otimes I_m\right)D_2\left[(D_1F)'\right] & \left(\bar{\theta}'\otimes I_m\right)D_3\left[(D_1F)'\right] & (D_1F)'\\ \left(\bar{\theta}'\otimes I_{n_\xi}\right)D_1\left[(D_2F)'\right] & \left(\bar{\theta}'\otimes I_{n_\xi}\right)D_2\left[(D_2F)'\right] & \left(\bar{\theta}'\otimes I_{n_\xi}\right)D_3\left[(D_2F)'\right] & (D_2F)'\\ \left(\bar{\theta}'\otimes I_m\right)D_1\left[(D_3F)'\right] & \left(\bar{\theta}'\otimes I_m\right)D_2\left[(D_3F)'\right] & \left(\bar{\theta}'\otimes I_m\right)D_3\left[(D_3F)'\right] & (D_3F)'\\ D_1F & D_2F & D_3F & 0\end{bmatrix}$$

has ji blocks of the form

$$\mathcal{H}_{ji}\left(\theta_t'F(y_t,\xi_t;y_{t-1})\right)=D_j\left\{\left[D_i\left(\theta_t'F(y_t,\xi_t;y_{t-1})\right)\right]'\right\}$$

evaluated at the optimal steady state, where the indices $j,i=1,...4$, refer respectively to the vectors $y_t,\xi_t,y_{t-1},\theta_t$, and where n_ξ is the dimension of the vector ξ_t. This approximation involves neither a constant term nor a first-order term in $\tilde{\theta}_t$, as we use the fact that $F\left(\bar{y},\bar{\xi};\bar{y}\right)=0$ in the optimal steady state. To obtain the derivatives $\mathcal{H}_{ji}\left(\theta_t'F(y_t,\xi_t;y_{t-1})\right)$, for $j,i=1,2,3$, we use

$$\begin{aligned}\mathcal{H}_{ji}\left(\theta_t'F(y_t,\xi_t;y_{t-1})\right) &= D_j\left\{\left[\theta_t'D_iF(y_t,\xi_t;y_{t-1})\right]'\right\}=D_j\left\{\left[D_iF(y_t,\xi_t;y_{t-1})\right]'\theta_t\right\}\\ &= D_j\left\{vec\left(\left[D_iF(y_t,\xi_t;y_{t-1})\right]'\theta_t\right)\right\}=D_j\left\{(\theta_t'\otimes I)\cdot vec\left(\left[D_iF(y_t,\xi_t;y_{t-1})\right]'\right)\right\}\\ &= (\theta_t'\otimes I)\cdot D_j\left\{\left[D_iF(y_t,\xi_t;y_{t-1})\right]'\right\}.\end{aligned}$$

To obtain second-order approximation of $\Theta_{t-1}'g\left(y_{t-1},\xi_{t-1};y_t\right)$, we proceed similarly, replacing $F(\ )$ with $g(\ )$ and θ_t with Θ_{t-1}.

The second-order approximation of the Lagrangian is thus given by

$$
\begin{aligned}
\mathcal{L}_{t_0} \;=\; & E_{t_0}\Bigg\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\Bigg(\pi\left(\bar{y},\bar{\xi}\right)+\left[D_1\pi,D_2\pi\right]\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\end{bmatrix}+\frac{1}{2}\left[\tilde{y}_t',\tilde{\xi}_t'\right]\begin{bmatrix} D_1\left[(D_1\pi)'\right] & D_2\left[(D_1\pi)'\right]\\ D_1\left[(D_2\pi)'\right] & D_2\left[(D_2\pi)'\right]\end{bmatrix}\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\end{bmatrix}\\
& +\bar{\theta}'\left[D_1F,D_2F,D_3F\right]\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\\ \tilde{y}_{t-1}\end{bmatrix}+\frac{1}{2}\left[\tilde{y}_t',\tilde{\xi}_t',\tilde{y}_{t-1}',\tilde{\theta}_t'\right]\mathcal{H}\left(\bar{\theta}'F\right)\begin{bmatrix}\tilde{y}_t\\ \tilde{\xi}_t\\ \tilde{y}_{t-1}\\ \tilde{\theta}_t\end{bmatrix}\\
& +\beta^{-1}\bar{\Theta}'\left[D_1g,D_2g,D_3g\right]\begin{bmatrix}\tilde{y}_{t-1}\\ \tilde{\xi}_{t-1}\\ \tilde{y}_t\end{bmatrix}+\frac{\beta^{-1}}{2}\left[\tilde{y}_{t-1}',\tilde{\xi}_{t-1}',\tilde{y}_t',\tilde{\Theta}_{t-1}'\right]\mathcal{H}\left(\bar{\Theta}'g\right)\begin{bmatrix}\tilde{y}_{t-1}\\ \tilde{\xi}_{t-1}\\ \tilde{y}_t\\ \tilde{\Theta}_{t-1}\end{bmatrix}\Bigg)\Bigg\}\\
& +\mathcal{O}(\epsilon^3)
\end{aligned}
$$

Expanding this expression, recognizing that each quadratic term is a scalar so that it is equal to its transpose, and using the definitions of the matrices $\bar{A}$, $\bar{I}$ and $\bar{C}$ in (2.8), and of S, R and the matrix polynomial $B\left(L\right)$ in (2.16), we can rewrite the Lagrangian as

$$
\begin{aligned}
\mathcal{L}_{t_0} \;=\; & E_{t_0}\Bigg\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\Bigg(\pi\left(\bar{y},\bar{\xi}\right)+\left[D_2\pi+\bar{\theta}'\left(D_2F\right)\right]\tilde{\xi}_t+\beta^{-1}\bar{\Theta}'\left(D_2g\right)\tilde{\xi}_{t-1}\\
& +\frac{1}{2}\tilde{\xi}_t'\left(D_2\left[(D_2\pi)'\right]+\left(\bar{\theta}'\otimes I_{n_\xi}\right)D_2\left[(D_2F)'\right]\right)\tilde{\xi}_t+\frac{\beta^{-1}}{2}\tilde{\xi}_{t-1}'\left(\bar{\Theta}'\otimes I_{n_\xi}\right)D_2\left[(D_2g)'\right]\tilde{\xi}_{t-1}\\
& +\left[D_1\pi+\bar{\varphi}'\left(\bar{A}-\beta^{-1}\bar{I}\right)\right]\tilde{y}_t\\
& +\frac{1}{2}\left[\tilde{y}_t'S\tilde{y}_t+\tilde{y}_t'R\tilde{y}_{t-1}+\tilde{y}_{t-1}'R'\tilde{y}_t\right]+\tilde{y}_t'\left[B\left(L\right)\tilde{\xi}_{t+1}\right]+\left[\tilde{\varphi}_{t+1}'\bar{A}-\beta^{-1}\tilde{\varphi}_t'\bar{I}\right]\tilde{y}_t+\tilde{\varphi}_{t+1}'\bar{C}\tilde{\xi}_{t+1}\Bigg)\Bigg\}\\
& +\beta^{-1}\tilde{\varphi}_{t_0}'\bar{A}\tilde{y}_{t_0-1}+\tilde{\theta}_{t_0}'\left(D_2F\right)\tilde{\xi}_{t_0}+\beta^{-1}\tilde{\Theta}_{t_0-1}'\left(D_2g\right)\tilde{\xi}_{t_0-1}\\
& +\beta^{-1}\bar{\varphi}'\bar{A}\tilde{y}_{t_0-1}+\frac{1}{2}\tilde{y}_{t_0-1}'\left\{\left(\bar{\theta}'\otimes I_m\right)D_3\left[(D_3F)'\right]+\beta^{-1}\left(\bar{\Theta}'\otimes I_m\right)D_1\left[(D_1g)'\right]\right\}\tilde{y}_{t_0-1}\\
& +\tilde{y}_{t_0-1}'\left\{\left(\bar{\theta}'\otimes I_m\right)D_2\left[(D_3F)'\right]\tilde{\xi}_{t_0}+\beta^{-1}\left(\bar{\Theta}'\otimes I_m\right)D_2\left[(D_1g)'\right]\tilde{\xi}_{t_0-1}\right\}+\mathcal{O}(\epsilon^3).
\end{aligned}
$$

In deriving this expression, we use the properties of the transposes discussed in the proof of Lemma 14, so that, e.g.,

$$\left(\bar{\theta}'\otimes I_{n_\xi}\right)D_1\left[(D_2F)'\right]=\left\{\left(\bar{\theta}'\otimes I_m\right)D_2\left[(D_1F)'\right]\right\}'.$$

In this approximation of the Lagrangian, we note that the terms in the first two lines are either a constant, or functions only of exogenous disturbances, so that they are independent of the path of endogenous variables (hence of the policy chosen at date t_0). Furthermore, the steady-state condition (1.12) implies that the term in square brackets in the third line is equal to zero, thereby cancelling the terms that are linear in $\tilde{y}_t$. In addition, the last two rows involve terms independent of any policy chosen at date t_0, and terms of third order or smaller. It follows that the Lagrangian can be rewritten as

$$\begin{aligned}\mathcal{L}_{t_0} &= E_{t_0}\left\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left[\tilde{y}_t'\left(\frac{1}{2}S\tilde{y}_t+R\tilde{y}_{t-1}+\left[B\left(L\right)\tilde{\xi}_{t+1}\right]+\left(\bar{A}-\beta^{-1}\bar{I}L\right)'\tilde{\varphi}_{t+1}\right)+\tilde{\varphi}_{t+1}'\bar{C}\tilde{\xi}_{t+1}\right]\right\}\\ &\quad+\beta^{-1}\tilde{\varphi}_{t_0}'\bar{A}'\tilde{y}_{t_0-1}+\tilde{\theta}_{t_0}'\left(D_2F\right)\tilde{\xi}_{t_0}+\beta^{-1}\tilde{\Theta}_{t_0-1}'\left(D_2g\right)\tilde{\xi}_{t_0-1}+tip+\mathcal{O}(\epsilon^3)\end{aligned}$$

where tip denotes terms independent of policy chosen at date t_0.

C.2 Perturbation of the Lagrangian Around the Optimal Steady State

As discussed in section 2.2.1, we consider a perturbation of the path $\{y_t\}$ from its optimal steady state equilibrium, maintaining the other vectors $\xi_t, \theta_t, \Theta_t$ unchanged. The perturbation is such that $\tilde{y}_t^{\varepsilon}=\tilde{y}_t$ for all $t<\tau$, and an arbitrary date $\tau\geq t_0$, while $\tilde{y}_{\tau+i}^{\varepsilon}=\tilde{y}_{\tau+i}+\varepsilon y_i$ for all $i\geq 0$. Here $\{y_i\}$ is a bounded sequence of vectors of dimension m, not all equal to zero, such that

$$\bar{I}y_0=0,\qquad \bar{A}y_i-\bar{I}y_{i+1}=0\quad\text{for all } i\geq 0, \tag{C.1}$$

and ε is a random quantity, the value of which is determined only at date τ. (In the case that ε is determined at date $\tau=t_0$, then it turns out that ε is a deterministic quantity).

The perturbed path $\{\tilde{y}_t^{\varepsilon}\}$ remains consistent with the linearized structural equations (2.8)–(2.9) as long as the random variable ε has conditional mean zero as of date $\tau-1$ (regardless of the state of the world at that date). If we let ε be some mean-zero bounded random variable with unit variance multiplied by a scale factor $|\varepsilon|$, then for any small enough value of $|\varepsilon|$, then this also approximates a feasible perturbation under the exact structural relations (1.2)–(1.4), up to an error of order $\mathcal{O}(|\varepsilon|^2)$.

In the perturbed equilibrium, the Lagrangian is equal to

$$\begin{aligned}\mathcal{L}_{t_0}^{\varepsilon} &= E_{t_0}\left\{\sum_{t=t_0}^{\tau-1}\beta^{t-t_0}\left[\tilde{y}_t'\left(\frac{1}{2}S\tilde{y}_t+R\tilde{y}_{t-1}+\left[B\left(L\right)\tilde{\xi}_{t+1}\right]+\left(\bar{A}-\beta^{-1}\bar{I}L\right)'\tilde{\varphi}_{t+1}\right)+\tilde{\varphi}_{t+1}'\bar{C}\tilde{\xi}_{t+1}\right]\right\}\\ &\quad+E_{t_0}\left\{\beta^{t-t_0}\left(\tilde{y}_\tau+\varepsilon y_0\right)'R\tilde{y}_{\tau-1}\right\}\\ &\quad+E_{t_0}\left\{\sum_{i=0}^{\infty}\beta^{\tau+i-t_0}\left[\left(\tilde{y}_{\tau+i}+\varepsilon y_i\right)'\left(\frac{1}{2}S\left(\tilde{y}_{\tau+i}+\varepsilon y_i\right)+\beta R'\left(\tilde{y}_{\tau+i+1}+\varepsilon y_{i+1}\right)+\left[B\left(L\right)\tilde{\xi}_{\tau+i+1}\right]\right.\right.\right.\\ &\quad\left.\left.\left.+\left(\bar{A}-\beta^{-1}\bar{I}L\right)'\tilde{\varphi}_{\tau+i+1}\right)+\tilde{\varphi}_{\tau+i+1}'\bar{C}\tilde{\xi}_{\tau+i+1}\right]\right\}\\ &\quad+\beta^{-1}\tilde{\varphi}_{t_0}'\bar{A}'\tilde{y}_{t_0-1}+\tilde{\theta}_{t_0}'\left(D_2F\right)\tilde{\xi}_{t_0}+\beta^{-1}\tilde{\Theta}_{t_0-1}'\left(D_2g\right)\tilde{\xi}_{t_0-1}+tip+\mathcal{O}(|\epsilon,\varepsilon|^3).\end{aligned}$$

The increase in the Lagrangian due to the perturbation can thus be expressed as

$$\begin{aligned}\mathcal{L}_{t_0}^{\varepsilon}-\mathcal{L}_{t_0} &= E_{t_0}\left\{\sum_{i=0}^{\infty}\beta^{\tau+i-t_0}\left(\frac{1}{2}\left[y_i'Sy_i+y_{i+1}'\beta Ry_i+y_i'\beta R'y_{i+1}\right]\varepsilon^2\right.\right.\\ &\quad\left.\left.+y_i'\left[S\tilde{y}_{\tau+i}+\beta R'\tilde{y}_{\tau+i+1}+R\tilde{y}_{\tau+i-1}+\left(\bar{A}-\beta^{-1}\bar{I}L\right)'\tilde{\varphi}_{\tau+i+1}+B\left(L\right)\tilde{\xi}_{\tau+i+1}\right]\varepsilon\right)\right\}\\ &\quad+\mathcal{O}(|\epsilon,\varepsilon|^3).\end{aligned}$$

Since the linearized first-order conditions (2.16) imply that the term in square brackets in the second line is of second order, this term becomes of third order once multiplied by ε, so that the increase in the Lagrangian due to the perturbation reduces to

$$\mathcal{L}_{t_0}^{\varepsilon} - \mathcal{L}_{t_0} = \frac{1}{2} \sum_{i=0}^{\infty} \beta^{\tau+i-t_0} \left[y_i' S y_i + y_{i+1}' \beta R y_i + y_i' \beta R' y_{i+1} \right] |\varepsilon|^2 + \mathcal{O}(|\epsilon, \varepsilon|^3).$$

D Real Kronecker Canonical Form

The following theorem adapts Gantmacher's (1959) proof of Theorem 3 (Chap. 12, vol. 2) to the case of a real Kronecker canonical form of regular matrix pencils.

Theorem 7 *Real Kronecker canonical form. Consider the matrix pencil $A - \mu \hat{I}$, with $A, \hat{I} \in \mathbb{R}^{n \times n}$. Suppose that its characteristic polynomial* $\det[A - \mu \hat{I}]$*, of order $n-k$, for $0 \le k \le n$, is not identically zero. Then there exist non-singular matrices $\tilde{U}, \tilde{V} \in \mathbb{R}^{n \times n}$ satisfying*

$$\tilde{U} \left(A - \mu \hat{I} \right) \tilde{V} = \begin{bmatrix} I_k & 0 \\ 0 & \tilde{G} \end{bmatrix} - \mu \begin{bmatrix} \tilde{H} & 0 \\ 0 & I_{n-k} \end{bmatrix},$$

where $\tilde{H} \in \mathbb{R}^{k \times k}$ is a nilpotent matrix of the Jordan form (i.e., with ones on the first super diagonal and zeros everywhere else), and $\tilde{G} \in \mathbb{R}^{(n-k)\times(n-k)}$ is a block-diagonal matrix of the real Jordan form. Each of the diagonal blocks of $\tilde{G}$ is either of the form

$$\begin{bmatrix} \mu_i & 1 & 0 & & 0 \\ 0 & \mu_i & 1 & \ddots & \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & 1 \\ 0 & & & 0 & \mu_i \end{bmatrix} \tag{D.1}$$

where $\mu_i \in \mathbb{R}$, or

$$\begin{bmatrix} M_i & I_2 & 0 & & 0 \\ 0 & M_i & I_2 & \ddots & \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & I_2 \\ 0 & & & 0 & M_i \end{bmatrix} \tag{D.2}$$

where the submatrices M_i are of the form

$$M_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$$

with $\alpha_i, \beta_i \in \mathbb{R}$. Furthermore, the $n-k$ finite eigenvalues (including their algebraic multiplicity) of the matrix pencil $A - \mu \hat{I}$ are equal to eigenvalues (including their algebraic multiplicity) of the matrix pencil $\tilde{G} - \mu I$.

Proof. Note that $P(\mu) = (A - \mu\hat{I})$ is a regular pencil since A, $\hat{I}$ are square matrices and $\det[A - \mu\hat{I}]$ is not identically zero. Then there exists a number $c \in \mathbb{R}$ such that $A_1 \equiv A - c\hat{I} \in \mathbb{R}^{n\times n}$ satisfies $|A_1| \neq 0$. We represent the given pencil in the form $P(\mu) = A - c\hat{I} - (\mu - c)\hat{I} = A_1 - (\mu - c)\hat{I}$, and multiply it on the left by A_1^{-1} to obtain $A_1^{-1}P(\mu) = I - (\mu - c)A_1^{-1}\hat{I}$. By similarity transformation into a real Jordan canonical form (Laub (2005), p. 83), there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that $A_1^{-1}\hat{I} = T\{J_0, \tilde{J}\}T^{-1}$ where $\{J_0, \tilde{J}\} \in \mathbb{R}^{n\times n}$ denotes a block diagonal matrix in which J_0 is a nilpotent Jordan matrix, and $\tilde{J} = \{J_1, J_2, ..., J_q\}$ is a real, block diagonal matrix which satisfies $\det\tilde{J} \neq 0$, with each block being of the form (D.1) in the case that μ_i are real eigenvalues of $A_1^{-1}\hat{I}$, and of the form (D.2) in the case of complex conjugate eigenvalues $\mu_i = \alpha_i \pm \beta i$ of $A_1^{-1}\hat{I}$.

Using this we can put the pencil in the form

$$A_1^{-1}P(\mu) = I - (\mu - c)\,T\{J_0, \tilde{J}\}T^{-1}$$

or equivalently

$$T^{-1}A_1^{-1}P(\mu)\,T = \{I + cJ_0 - \mu J_0, I + c\tilde{J} - \mu\tilde{J}\}.$$

Multiplying on the right by $\{(I + cJ_0)^{-1}, \tilde{J}^{-1}\}$ we have[25]

$$T^{-1}A_1^{-1}P(\mu)\,T\{(I + cJ_0)^{-1}, \tilde{J}^{-1}\} = \{I - \mu(I + cJ_0)^{-1}J_0, (\tilde{J}^{-1} + cI) - \mu I\}.$$

By a similarity transformation, we can write $(I + cJ_0)^{-1}J_0 = T_0\tilde{H}T_0^{-1}$ where T_0 is a real matrix which satisfies $|T_0| \neq 0$ and where $\tilde{H} = \left\{H^{(u_1)}, H^{(u_2)}, ...H^{(u_s)}\right\}$ is a Jordan matrix in which the submatrices $H^{(u)}$ of order u are have ones on the first superdiagonal and zeroes in all other elements. (This is because J_0, and hence $(I + cJ_0)^{-1}J_0$, are nilpotent and that all eigenvalues of a nilpotent matrix are always 0).

Thus by multiplying the matrix pencil above by $\left\{T_0^{-1}, I\right\}$ on the left and $\{T_0, I\}$ on the right, we get

$$\begin{aligned}\tilde{U}(A - \mu\hat{I})\tilde{V} &= \{T_0^{-1}T_0 - \mu T_0^{-1}J_0(I + cJ_0)^{-1}T_0, (\tilde{J}^{-1} + cI) - \mu I\} \\ &= \{I - \mu\tilde{H}, \tilde{G} - \mu I\}\end{aligned}$$

where $\tilde{U} \equiv \{T_0^{-1}, I\}T^{-1}A_1^{-1}$, $\tilde{V} \equiv T\{(I + cJ_0)^{-1}, \tilde{J}^{-1}\}\{T_0, I\}$, and $\tilde{G} \equiv \tilde{J}^{-1} + cI$ are real matrices. Given that the matrices $\tilde{U}$ and $\tilde{V}$ are non-singular, the matrix pencils $(A - \mu\hat{I})$ and $\{I - \mu\tilde{H}, \tilde{G} - \mu I\}$ are strictly equivalent (see Definition 1, Gantmacher, 1959, Chap. 12). It follows from Theorem 2 of Gantmacher (1959, Chap. 12) that these two matrix pencils have the same ("finite" and "infinite") elementary divisors. Since all k infinite elementary divisors of $(A - \mu\hat{I})$ are associated with $I - \mu\tilde{H}$, and the $n - k$ finite elementary divisors are associated with $\tilde{G} - \mu I$, the $n - k$ finite eigenvalues (including their algebraic multiplicity) of the matrix pencil $A - \mu\hat{I}$ are equal to the eigenvalues (including their algebraic multiplicity) of the matrix pencil $\tilde{G} - \mu I$. Thus $\tilde{H}$ is of dimensions $k \times k$ while $\tilde{G}$ is $(n - k) \times (n - k)$. ∎

[25] Note that $(I + cJ_0 - \mu J_0)(I + cJ_0)^{-1} = I - \mu J_0(I + cJ_0)^{-1} = I - \mu\left(J_0^{-1}\right)^{-1}(I + cJ_0)^{-1} = I - \mu\left((I + cJ_0)J_0^{-1}\right)^{-1} = I - \mu\left(J_0^{-1} + cI\right)^{-1} = I - \mu\left(J_0^{-1}(I + cJ_0)\right)^{-1} = I - \mu(I + cJ_0)^{-1}J_0$.

We have thus shown that for any real regular pencil $A - \mu \hat{I}$, there exist real matrices $\tilde{U}, \tilde{V}$ such that

$$\tilde{U} A \tilde{V} = \begin{bmatrix} I & 0 \\ 0 & \tilde{G} \end{bmatrix}, \qquad \tilde{U} \hat{I} \tilde{V} = \begin{bmatrix} \tilde{H} & 0 \\ 0 & I \end{bmatrix}$$

where $\tilde{G}$ is an invertible matrix of the (real) Jordan form and $\tilde{H}$ is an nilpotent matrix of the Jordan form.

E Target Criterion in Model of Section 4

E.1 Some Details on the Model

In this model, each household seeks to maximize its lifetime utility

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\tilde{u}\left(C_t; \xi_t\right) - \int_0^1 \tilde{v}\left(h_t\left(j\right); \xi_t\right) dj \right]$$

where $C_t \equiv \left[\int_0^1 c_t\left(i\right)^{(\theta-1)/\theta} \right]^{\theta/(\theta-1)}$ is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods, with an elasticity of substitution $\theta > 1$, and $h_t(j)$ is the quantity supplied of labor of type j. Each differentiated good is supplied by a single monopolistically competitive producer who uses labor of a particular type. The representative household supplies all types of labor. The preference functions are assumed to be of the form

$$\tilde{u}\left(C_t; \xi_t\right) \equiv \frac{C_t^{1-\tilde{\sigma}^{-1}} \bar{C}_t^{\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}}, \qquad \tilde{v}\left(h_t\left(j\right); \xi_t\right) \equiv \frac{\lambda}{1+\nu} h_t\left(j\right)^{1+\nu} \bar{h}_t^{-\nu}$$

where $\tilde{\sigma}, \nu > 0$, and $\left\{\bar{C}_t, \bar{h}_t\right\}$ are exogenous disturbance processes included in the vector ξ_t, which is assumed to be bounded up to some date T, and constant for all $t > T$.

Each specialized good is produced according to the production function

$$y_t\left(i\right) = A_t h_t\left(i\right)^{1/\phi}$$

where A_t is an exogenously varying technology factor (also included in the vector ξ_t), and $\phi > 1$. Implicitly, we consider other factors of production such as the capital stock as being constant or exogenously varying. Aggregate output Y_t in turn relates to consumption according to

$$Y_t = C_t + G_t$$

where G_t denotes exogenous government demand for the composite good and is also included in the vector ξ_t.

As Benigno and Woodford (2005) show, the utility of the representative household, which is also the policymaker's welfare objective function, can be expressed in the form (1.1) where the period t utility is

$$\pi\left(y_t, \xi_t\right) = U(Y_t, \Delta_t; \xi_t) \equiv u\left(Y_t; \xi_t\right) - v\left(Y_t; \xi_t\right) \Delta_t, \tag{E.1}$$

where

$$u(Y_t, \xi_t) \equiv \tilde{u}(Y_t - G_t, \xi_t), \qquad v(Y_t; \xi_t) \equiv \tilde{v}((Y_t/A_t)^{\phi}; \xi_t)$$

express utility as functions of aggregate output and have the properties $\sigma^{-1} \equiv -u_{YY}\bar{Y}/u_Y = \tilde{\sigma}^{-1}\bar{Y}/\bar{C} > 0$, $\omega \equiv v_{YY}\bar{Y}/v_Y = v_Y\bar{Y}/v - 1 > 0$, and

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t}\right)^{-\theta(1+\omega)} di \geq 1$$

is measure of price dispersion at date t, where $p_t(i)$ denotes the price of individual good i and $P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta}\right]^{1/(1-\theta)}$ is a Dixit-Stiglitz price index.

The producers are wage takers on the labor market and choose their prices to maximize the present discounted value of future after-tax nominal profits. As in Calvo's (1983) model of staggered pricing, we assume that producers fix the prices of their goods for a random interval of time, and that a constant fraction $\alpha \in [0, 1)$ of prices remain unchanged in any given period. As shown in Benigno and Woodford (2005), since all suppliers that revise their prices in period t face the same problem, they all choose the same new price p_t^* satisfying the first-order condition

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{H_t}\right)^{\frac{1}{1+\omega\theta}} \tag{E.2}$$

where

$$\begin{aligned} H_t &\equiv E_t\left[\sum_{T=t}^{\infty} (\alpha\beta)^{T-t} (1 - \varsigma_T) \cdot u_Y(Y_T, \xi_T) \cdot Y_T \cdot (P_T/P_t)^{\theta-1}\right] \\ K_t &\equiv E_t\left[\sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{\theta\mu_T^w}{\theta - 1} \cdot v_Y(Y_T, \xi_T) \cdot Y_T \cdot (P_T/P_t)^{\theta(1+\omega)}\right] \end{aligned}$$

and $\varsigma_t \in [0, 1)$ is an exogenous tax rate on sales revenues and $\mu_t^w \geq 1$ is an exogenous markup factor on the labor market.

The price index in turn evolves according to a law of motion

$$P_t = \left[(1 - \alpha) p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{1/(1-\theta)}. \tag{E.3}$$

Combining (E.2) with (E.3) yields

$$\frac{1 - \alpha\Pi_t^{\theta-1}}{1 - \alpha} = \left(\frac{H_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}} \tag{E.4}$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. This expression is a short-run aggregate supply relation between inflation and output, given the current disturbances ξ_t and expected future inflation, output, and disturbances.

Using again (E.3), we can also obtain an expression for the evolution of the measure of price dispersion

$$\Delta_t = \alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)} + (1 - \alpha)\left(\frac{1 - \alpha\Pi_t^{\theta-1}}{1 - \alpha}\right)^{\frac{\theta(1+\omega)}{\theta-1}}. \tag{E.5}$$

We assume that the government imposes lump-sum taxes on households so as to guarantee its intertemporal solvency regardless of monetary policy actions. In addition, we abstract from monetary frictions that would generate a demand for money, and assume that the policymaker can control the riskless short-term nominal interest rate and that the lower bound on nominal interest rates never binds. While the optimal intertemporal allocation of households' expenditures determines period-t output as a function of expectations of future output, inflation and the nominal interest, this doesn't constitute a constraint on the policy problem as the central bank can always choose a nominal interest rate that satisfies this equation. As a result, the only relevant constraints facing the policymaker are given by (E.4) and (E.5). While the former prevents the central bank from simultaneously stabilizing inflation and output, the latter determines the evolution of the price dispersion, *i.e.*, a key source of welfare losses.

It will be convenient to rewrite the expressions for H_t, K_t in recursive form as in (4.2), (4.3), and to use (E.4) to substitute for the variable Π_t from the system. The resulting restrictions (E.5), (4.2)–(4.3) can then be expressed as in (1.2)–(1.3) where

$$F(y_t, \xi_t; y_{t-1}) \equiv -\Delta_t + \check{f}(\Delta_{t-1}, Z_t) \tag{E.6}$$
$$g(y_t, \xi_t; y_{t+1}) \equiv \check{g}(Y_t, \xi_t) - Z_t + \alpha\beta\Phi(Z_{t+1}) \tag{E.7}$$

where

$$\check{f}(\Delta_{t-1}, Z_t) \equiv \alpha\Delta_{t-1}\left[\frac{1}{\alpha} - \frac{1-\alpha}{\alpha}\left(\frac{H_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}\right]^{\frac{\theta(1+\omega)}{\theta-1}} + (1-\alpha)\left(\frac{H_t}{K_t}\right)^{\frac{\theta(1+\omega)}{1+\omega\theta}}$$

$$\check{g}(Y_t, \xi_t) = \begin{bmatrix} (1-\varsigma_t)\, u_Y(Y_t, \xi_t)\, Y_t \\ \frac{\theta\mu^w}{\theta-1} v_Y(Y_t, \xi_t)\, Y_t \end{bmatrix}, \qquad \Phi(Z_t) = \begin{bmatrix} \left(\frac{1}{\alpha} - \frac{1-\alpha}{\alpha}\left(\frac{H_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}\right) H_t \\ \left(\frac{1}{\alpha} - \frac{1-\alpha}{\alpha}\left(\frac{H_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}\right)^{\frac{\theta(1+\omega)}{\theta-1}} K_t \end{bmatrix}.$$

and the vector of endogenous variables is given by $y_t \equiv [Y_t, \Delta_t, H_t, K_t]'$, while $Z_t \equiv [H_t, K_t]'$ is a subset of the endogenous variables.

E.2 Steady State

We now show that an optimal steady state exists in which the inflation rate is zero ($\bar{\Pi} = 1$). The optimal steady state is described by constant vectors $(\bar{y}, \bar{\varphi}, \bar{\xi})$ satisfying (1.10)–(1.12), or, in this model,

$$-\bar{\Delta} + \check{f}(\bar{\Delta}, \bar{Z}) = 0 \tag{E.8}$$
$$\check{g}(\bar{Y}, \bar{\xi}) - \bar{Z} + \alpha\beta\Phi(\bar{Z}) = 0 \tag{E.9}$$
$$U_Y(\bar{Y}, \bar{\Delta}; \bar{\xi}) + \check{g}_Y(\bar{Y}, \bar{\xi})'\bar{\Theta} = 0 \tag{E.10}$$
$$U_\Delta(\bar{Y}, \bar{\Delta}; \bar{\xi}) + \left(\beta\check{f}_\Delta(\bar{\Delta}, \bar{Z}) - 1\right)\bar{\theta} = 0 \tag{E.11}$$
$$\check{f}_Z(\bar{\Delta}, \bar{Z})'\bar{\theta} + \left(\alpha\Phi_Z(\bar{Z})' - I_2\right)\bar{\Theta} = 0, \tag{E.12}$$

where $\bar{Z} = [\bar{H}, \bar{K}]'$ together with the steady-state versions of equation (E.4). We proceed by conjecturing that the solution involves $\bar{\Pi} = 1$, and showing that a solution can be constructed that satisfies all of the equations just listed.

We first observe that (E.4) implies that a steady state with $\bar{\Pi} = 1$ must satisfy $\bar{H} = \bar{K}$. Given this, (E.8) requires that $\bar{\Delta} = 1$, so that there is zero price dispersion. Condition (E.9) holds as well if and only if $\bar{Y}$ is the output level implicitly defined by

$$(1 - \varsigma)\, u_Y\left(\bar{Y}, \bar{\xi}\right) = \frac{\theta \bar{\mu}^w}{\theta - 1} v_Y\left(\bar{Y}, \bar{\xi}\right), \tag{E.13}$$

and $\bar{H} = \bar{K} = (1 - \alpha\beta)^{-1} \frac{\theta \mu^w}{\theta - 1} v_Y\left(\bar{Y}, \bar{\xi}\right) \bar{Y}$.

Because $\check{f}_Z\left(1, \bar{Z}\right) = 0$, (E.12) reduces to the eigenvector equation

$$\left(\alpha \Phi_Z(\bar{Z})' - I_2\right) \bar{\Theta} = 0 \tag{E.14}$$

where

$$\Phi_Z(\bar{Z}) = \begin{bmatrix} \frac{1}{\alpha} + \frac{\theta(\alpha-1)(\omega+1)}{\alpha(\theta\omega+1)} & -\frac{(\theta-1)(\alpha-1)}{\alpha(\theta\omega+1)} \\ \frac{\theta(\alpha-1)(\omega+1)}{\alpha(\theta\omega+1)} & 1 - \frac{\theta(\alpha-1)(\omega+1)}{\alpha(\theta\omega+1)} \end{bmatrix}.$$

Since $\Phi_Z(\bar{Z})'$ has an eigenvector $[-1, 1]'$, with eigenvalue $1/\alpha$, (E.14) is satisfied if and only if $\bar{\Theta}_2 = -\bar{\Theta}_1$.

Conditions (E.10)–(E.11) in turn allow us to determine the values $\bar{\theta}, \bar{\Theta}$. Given that $\check{f}_\Delta\left(\bar{\Delta}, \bar{Z}\right) = \alpha$ and

$$\check{g}_Y(\bar{Y}, \bar{\xi}) = \begin{bmatrix} (1 - \varsigma)\left(u_{YY}\bar{Y} + u_Y\right) \\ \frac{\theta \mu^w}{\theta - 1}\left(v_{YY}\bar{Y} + v_Y\right) \end{bmatrix}$$

we obtain

$$\bar{\theta} = -\frac{v\left(\bar{Y}, \bar{\xi}\right)}{1 - \alpha\beta}, \qquad \bar{\Theta}_1 = -\bar{\Theta}_2 = \frac{\Psi}{(1 - \varsigma)\left(\omega + \sigma^{-1}\right)},$$

where

$$\Psi \equiv 1 - \frac{\theta - 1}{\theta} \frac{1 - \varsigma}{\mu^w}$$

is a measure of the degree of inefficiency of the steady-state output level $\bar{Y}$. Here we use the fact that $U_Y = u_Y - v_Y = \Psi u_Y$ and $u_{YY} - v_{YY} = -\frac{u_{YY}}{\bar{Y}}\left(\omega\left(1 - \Psi\right) + \sigma^{-1}\right)$. (The first of these equations explains our interpretation of Ψ as a measure of the degree of inefficiency: a positive value of Ψ indicates that utility would be increased by raising $\bar{Y}$, maintaining zero price dispersion and hence an equal level of production of each of the differentiated goods.)

E.3 Canonical Decomposition of $\bar{A} - \mu \bar{I}$

Evaluating the derivatives of $F()$ and $g()$ at the steady state, we can then construct the matrices $\bar{A}$ and $\bar{I}$:

$$\bar{A} \equiv \begin{bmatrix} \beta D_3 F \\ D_1 g \end{bmatrix} = \begin{bmatrix} 0 & \alpha\beta & 0 & 0 \\ a_{21} & 0 & -1 & 0 \\ a_{31} & 0 & 0 & -1 \end{bmatrix}$$

$$\bar{I} \equiv \begin{bmatrix} -\beta D_1 F \\ -D_3 g \end{bmatrix} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 0 & 0 & -\alpha\beta + \frac{\beta(1-\alpha)(\theta-1)}{1+\omega\theta} & -\frac{\beta(1-\alpha)(\theta-1)}{1+\omega\theta} \\ 0 & 0 & \frac{\beta(1-\alpha)\theta(1+\omega)}{1+\omega\theta} & -\alpha\beta - \frac{\beta(1-\alpha)\theta(1+\omega)}{1+\omega\theta} \end{bmatrix},$$

where

$$a_{21} = (1-\varsigma)\, u_Y \left(1-\sigma^{-1}\right), \qquad a_{31} = (1-\varsigma)\, u_Y \,(1+\omega)\,.$$

The matrix pencil $\bar{A} - \mu \bar{I}$ pencil satisfies Assumption 2. We first determine the minimal degree associated with the matrix pencil $\bar{A} - \mu \bar{I}$. Following Gantmacher (1959, chap. 12, p. 30), the *minimal degree* of the matrix pencil $\bar{A} - \mu \bar{I}$ is the least value of the index l for which the rank of the matrix

$$M_l = \overbrace{\begin{bmatrix} \bar{A} & 0 & \cdots & 0 \\ -\bar{I} & \bar{A} & & \vdots \\ 0 & -\bar{I} & \ddots & 0 \\ \vdots & & \ddots & \bar{A} \\ 0 & 0 & \cdots & -\bar{I} \end{bmatrix}}^{l+1}$$

satisfies

$$rank\,(M_l) < (l+1)\, m.$$

We observe that $\text{rank}(M_0) = 4 = (0+1)\,4$, $\text{rank}(M_1) = 8 = (1+1)\,4$, but that $\text{rank}(M_2) = 11 < (2+1)\,4 = 12$. The minimal degree of the matrix pencil $\bar{A} - \mu \bar{I}$ is therefore $\tilde{n} = 2$, so that $q = n - \tilde{n} = 1$.

It follows from Lemma 3 that there exist nonsingular matrices P and Q of dimensions 3×3 and 4×4 respectively that satisfy

$$(\bar{A} - \mu \bar{I})' = Q \begin{bmatrix} \mu & 0 & 0 \\ 1 & \mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B_2' - \mu J_2' \end{bmatrix} P.$$

These matrices are given by

$$Q^{-1} = \begin{bmatrix} 1 & 0 & a_{21} & a_{31} \\ \beta\,(1+\alpha) & 0 & q_{23} & q_{24} \\ \alpha\beta^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & \beta a_{21}\,(\alpha+1) - q_{23} & \beta a_{31}\,(\alpha+1) - q_{24} \\ 0 & \alpha\beta^2 a_{21} & \alpha\beta^2 a_{31} \\ \beta & 0 & 0 \end{bmatrix}$$

where $\text{rank}(Q^{-1}) = 4$ and $\text{rank}(P) = 3$, and q_{23}, q_{24} satisfy $q_{23} - q_{24} = \alpha\beta\,(a_{21} - a_{31})\,.$ We also have $B_2 = \alpha$ and $J_2 = 1$.

E.4 Target Variables

When evaluating the second derivatives of $F\,()$, $g\,()$ and the objective function $\pi\,()$ at the optimal steady state, the target variables τ_t and the target values τ_t^* are given by

$$\begin{aligned} \tau_t &= -[S\tilde{y}_t + R\tilde{y}_{t-1} + \beta R' E_t \tilde{y}_{t+1}] \\ \tau_t^* &= \left\{ D_2\left[(D_1\pi)'\right] + \left(\bar{\Theta}' \otimes I_4\right) D_2\left[(D_1 g)'\right] \right\} \tilde{\xi}_t, \end{aligned}$$

where the matrices S and R reduce to

$$\begin{aligned} S &\equiv D_1\left[(D_1\pi)'\right] + \bar{\theta} D_1\left[(D_1 F)'\right] + \left(\bar{\Theta}'\otimes I_4\right)\left\{D_1\left[(D_1 g)'\right] + \beta^{-1} D_3\left[(D_3 g)'\right]\right\} \\ &= \begin{bmatrix} U_{YY} + \bar{\Theta}'\breve{g}_{YY} & U_{Y\Delta} & 0_{1\times 2} \\ U_{\Delta Y} & 0 & 0_{1\times 2} \\ 0 & 0 & \bar{\theta}\check{f}_{ZZ}\left(\bar{\Delta},\bar{Z}\right) + \left(\bar{\Theta}'\otimes I_2\right)\alpha\Phi_{ZZ} \end{bmatrix}, \\ R &\equiv \bar{\theta} D_3\left[(D_1 F)'\right] = \begin{bmatrix} 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 2} \\ 0_{2\times 1} & \bar{\theta}\check{f}_{Z\Delta}\left(\bar{\Delta},\bar{Z}\right) & 0_{2\times 2} \end{bmatrix}. \end{aligned}$$

Here, we use properties of the second derivatives of $F()$ and $g()$: $D_3\left[(D_3 F)'\right] = 0$, $D_2\left[(D_1 F)'\right] = 0$, $D_2\left[(D_3 F)'\right] = 0$, $D_1\left[(D_3 g)'\right] = 0$, $D_2\left[(D_3 g)'\right] = 0$, and

$$\left(\bar{\Theta}'\otimes I_4\right) D_1\left[(D_1 g)'\right] = \begin{bmatrix} \bar{\Theta}'\breve{g}_{YY} & 0 \\ 0 & 0_{3\times 3} \end{bmatrix}, \quad \left(\bar{\Theta}'\otimes I_4\right) D_2\left[(D_1 g)'\right] = \begin{bmatrix} \bar{\Theta}'\breve{g}_{Y\xi} \\ 0_{3\times n_\xi} \end{bmatrix},$$

where $\breve{g}_{YY} = \partial\breve{g}_Y(\bar{Y},\bar{\xi})/\partial Y$, $\breve{g}_{Y\xi} = \partial\breve{g}_Y(\bar{Y},\bar{\xi})/\partial\xi'$. Similarly, we use

$$\left(\bar{\Theta}'\otimes I_4\right) D_3\left[(D_3 g)'\right] = \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \left(\bar{\Theta}'\otimes I_2\right)\alpha\beta\Phi_{ZZ} \end{bmatrix}$$

where $\Phi_{ZZ} = \partial vec\left(\Phi_Z(\bar{Z})'\right)/\partial Z'$. Moreover, since

$$\left\{D_2\left[(D_1\pi)'\right] + \left(\bar{\Theta}'\otimes I_4\right) D_2\left[(D_1 g)'\right]\right\} = \begin{bmatrix} U_{Y\xi} + \bar{\Theta}'\breve{g}_{Y\xi} \\ U_{\Delta\xi} \\ 0_{2\times 1} \end{bmatrix},$$

we can express the target gaps as

$$\tau_t - \tau_t^* = -\begin{bmatrix} \left(U_{YY} + \bar{\Theta}'\breve{g}_{YY}\right)\tilde{Y}_t + U_{Y\Delta}\tilde{\Delta}_t + \left(U_{Y\xi} + \bar{\Theta}'\breve{g}_{Y\xi}\right)\tilde{\xi}_t \\ U_{\Delta Y}\tilde{Y}_t + \beta\check{f}_{Z\Delta}\left(\bar{\Delta},\bar{Z}\right)'\bar{\theta}E_t\tilde{Z}_{t+1} + U_{\Delta\xi}\tilde{\xi}_t \\ \left(\bar{\theta}\check{f}_{ZZ}\left(\bar{\Delta},\bar{Z}\right) + \left(\bar{\Theta}'\otimes I_2\right)\alpha\Phi_{ZZ}\right)\tilde{Z}_t + \bar{\theta}\check{f}_{Z\Delta}\left(\bar{\Delta},\bar{Z}\right)\tilde{\Delta}_{t-1} \end{bmatrix}.$$

Linearizing the relationship implicitly defining Y_t^*, (4.6), we have

$$U_{YY}\tilde{Y}_t^* + U_{Y\xi}\tilde{\xi}_t + \bar{\Theta}'\left(\breve{g}_{YY}\tilde{Y}_t^* + \breve{g}_{Y\xi}\tilde{\xi}_t\right) = 0,$$

where $\tilde{Y}_t^* = Y_t^* - \bar{Y}$. This allows us to rewrite the first element of $\tau_t - \tau_t^*$ as

$$\left(U_{YY} + \bar{\Theta}'\breve{g}_{YY}\right)\bar{Y}x_t + U_{Y\Delta}\tilde{\Delta}_t,$$

where $x_t = \log\left(Y_t/Y_t^*\right) = \left(\tilde{Y}_t - \tilde{Y}_t^*\right)/\bar{Y} + \mathcal{O}(\epsilon^2)$ measures the welfare-relevant output gap. The target gaps $\tau_t - \tau_t^*$ can further be simplified, by noting that a linear approximation of (E.4) yields

$$\pi_t = -\frac{1-\alpha}{(1+\omega\theta)\,\alpha\bar{K}}\left(\tilde{H}_t - \tilde{K}_t\right)$$

where $\pi_t \equiv \log \Pi_t$. Using this and the fact that $\bar{\Theta}_1 = -\bar{\Theta}_2$, we obtain

$$\begin{aligned} \check{f}_{ZZ}\left(\bar{\Delta}, \bar{Z}\right)\tilde{Z}_t &= -\frac{1}{\bar{K}}\theta\left(1+\omega\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\pi_t \\ \left(\bar{\Theta}' \otimes I_2\right)\alpha\Phi_{ZZ}\tilde{Z}_t &= \bar{\Theta}_1\theta\left(\omega+1\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\pi_t, \end{aligned}$$

so that the target gaps can be rewritten as

$$\tau_t - \tau_t^* = -\begin{bmatrix} \left(U_{YY} + \bar{\Theta}'\check{g}_{YY}\right)\bar{Y}x_t + U_{Y\Delta}\tilde{\Delta}_t \\ U_{\Delta Y}\tilde{Y}_t + \beta\check{f}_{Z\Delta}\left(\bar{\Delta}, \bar{Z}\right)'\bar{\theta}E_t\tilde{Z}_{t+1} + U_{\Delta\xi}\tilde{\xi}_t \\ \theta\left(\omega+1\right)\left(\left(\bar{\Theta}_1 - \frac{\bar{\theta}}{\bar{K}}\right)\pi_t - \frac{\bar{\theta}}{\bar{K}}\frac{(1-\alpha)}{(1+\theta\omega)}\tilde{\Delta}_{t-1}\right) \\ -\theta\left(\omega+1\right)\left(\left(\bar{\Theta}_1 - \frac{\bar{\theta}}{\bar{K}}\right)\pi_t - \frac{\bar{\theta}}{\bar{K}}\frac{(1-\alpha)}{(1+\theta\omega)}\tilde{\Delta}_{t-1}\right) \end{bmatrix}. \tag{E.15}$$

E.5 Optimal Target Criterion

Since $k_1 = 0$, the general optimal target criterion (3.20) reduces to

$$0 = z_t \equiv \begin{bmatrix} 1 & -\beta^{-1}L & \beta^{-2}L^2 \end{bmatrix}\hat{\tau}_{1t},$$

where

$$\hat{\tau}_{1t} = \begin{bmatrix} I_{m-q} & 0 \end{bmatrix}Q^{-1}\left(\tau_t - \tau_t^*\right).$$

Combining this with (E.15), we obtain

$$\begin{aligned} 0 &= \begin{bmatrix} 1-(1+\alpha)L+\alpha L^2 & 0 & a_{21} - q_{23}\beta^{-1}L & a_{31} - q_{24}\beta^{-1}L \end{bmatrix}\left(\tau_t - \tau_t^*\right) \\ &= \left(1-\alpha L\right)\left(1-L\right)\left(\left(U_{YY} + \bar{\Theta}'\check{g}_{YY}\right)\bar{Y}x_t + U_{Y\Delta}\tilde{\Delta}_t\right) \\ &\quad + \left(a_{21} - a_{31}\right)\left(1-\alpha L\right)\theta\left(\omega+1\right)\left(\left(\bar{\Theta}_1 - \frac{\bar{\theta}}{\bar{K}}\right)\pi_t - \frac{\bar{\theta}}{\bar{K}}\frac{(1-\alpha)}{(1+\theta\omega)}\tilde{\Delta}_{t-1}\right), \end{aligned} \tag{E.16}$$

using $(q_{23} - q_{24}) = \alpha\beta\left(a_{21} - a_{31}\right)$ to obtain the last equality.

Noting furthermore that a first-order approximation of (E.6) around the optimal steady state yields

$$0 = -\tilde{\Delta}_t + \check{f}_\Delta\left(\bar{\Delta}, \bar{Z}\right)\tilde{\Delta}_{t-1} + \check{f}_Z\left(\bar{\Delta}, \bar{Z}\right)\tilde{Z}_t$$

or

$$\left(1-\alpha L\right)\tilde{\Delta}_t = 0$$

for all t, we observe that the variable $\tilde{\Delta}_t$ drops out from the target criterion, so that (E.16) reduces to

$$0 = \left(1-\alpha L\right)\left[\left(U_{YY} + \bar{\Theta}'\check{g}_{YY}\right)\bar{Y}\left(1-L\right)x_t + \left(a_{21} - a_{31}\right)\theta\left(\omega+1\right)\left(\bar{\Theta}_1 - \frac{\bar{\theta}}{\bar{K}}\right)\pi_t\right],$$

or equivalently

$$0 = \left(1-\alpha L\right)\left(\pi_t + \phi\left(1-L\right)x_t\right),$$

where the weight ϕ on changes in output gap fluctuations is given by

$$\phi = \frac{\left(U_{YY} + \bar{\Theta}' \breve{g}_{YY}\right)\bar{Y}}{\left(a_{21} - a_{31}\right)\theta\left(\omega + 1\right)\left(\bar{\Theta}_1 - \frac{\bar{\theta}}{\bar{K}}\right)} = \frac{\omega + \sigma^{-1} + \Psi\left(1 - \sigma^{-1}\right) - \frac{\Psi\sigma^{-1}\left(\bar{Y}/\bar{C} - 1\right)}{\omega + \sigma^{-1}}}{\theta\left(\omega + \sigma^{-1} + \Psi\left(1 - \sigma^{-1}\right)\right)}. \qquad \text{(E.17)}$$

To obtain the second equality in (E.17), we use (E.13) and the properties of the preference functions $\frac{u_{YYY}\bar{Y}^2}{u_Y} = \sigma^{-1}\left(\sigma^{-1} + \bar{Y}/\bar{C}\right)$ and $\frac{v_{YYY}\bar{Y}^2}{v_Y} = \omega\left(\omega - 1\right)$. We thus obtain the representation (4.5) for the optimal target criterion, as stated in the text.