The Demographics of Innovation and Asset Returns: Extended Appendix

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August 2009

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Contents

1 Labor input as a composite service and human capital depreciation

This section provides a more elaborate model of the labor market, that reproduces the path of labor income over an agent's life, as postulated in equations (10) and (11). The main difference between the baseline model and the model of this section is that the labor income process results from general-equilibrium wage effects, rather than assumptions on agents' endowments of labor efficiency units.

To draw this distinction, we assume that workers' efficiency units are only affected by aging and experience. Specifically, workers endowments of labor efficiency units evolve deterministically over their life according to $h_{t,s} = \overline{h}\,(1+\delta)^{t-s}$. However, the innovation shocks u_t no longer have any effects on agents' endowment of labor efficiency units.

Assume moreover, that labor is not a homogenous service. Instead, the units of labor that enter the production function of final goods and intermediate goods are measured in terms of a composite service, which is a constant elasticity of substitution (CES) aggregator of the labor efficiency units provided by workers belonging to different cohorts. Specifically, one unit of (composite) labor is given by

$$L_t = \left(\sum_{s=-\infty}^{t} v_{t,s}^{\frac{1}{b}} \left(l_{t,s}\right)^{\frac{b-1}{b}} \right)^{\frac{b}{b-1}}, \tag{1.1}$$

where $l_{t,s}$ denotes the labor input of cohort s at time t, $v_{t,s} > 0$ controls the relative importance of this input and $b > 0$ is the elasticity of substitution. The production function of final goods continues to be given by (3) and it still takes one unit of the composite labor service to produce one unit of the intermediate good. Equation (1.1) captures the idea that different cohorts have different skills and hence they are imperfect substitutes in the production process. Next, we let

$$v_{t,s}^{\frac{1}{b}} \equiv (1-\phi)^{\left(\frac{b-1}{b}\right)} q_{t,s} h_{t,s}^{\frac{1}{b}}. \tag{1.2}$$

Before proceeding, we note that using (1.2) inside (1.1), recognizing that in equilibrium $l_{t,s} =$

$h_{t,s}$, and noting that $\sum_{s=-\infty}^{t} q_{t,s} h_{t,s} = 1$ implies that the aggregate supply of (composite) labor efficiency units is constant and equal to $(1-\phi)$.

Since labor inputs by agents belonging to different cohorts are imperfect substitutes, we need to solve for the equilibrium wage $w_{t,s}$ of each separate cohort. This process is greatly facilitated by first constructing a "wage index", i.e., taking a set of cohort-specific wages as given, and then minimizing (over cohort labor inputs) the cost of obtaining a single unit of the composite labor input. As is well established in the literature, this wage index for CES production functions is given by

$$\overline{w}_t = \left(\sum_{s=-\infty}^{t} v_{t,s} \left(w_{t,s}\right)^{1-b} \right)^{\frac{1}{1-b}} .$$

With this wage index at hand, the cohort-specific input demands for a firm demanding a total of L_t units of the composite good are given by

$$w_{t,s} = \overline{w}_t v_{t,s}^{\frac{1}{b}} \left(\frac{l_{t,s}}{L_t} \right)^{-\frac{1}{b}} . \tag{1.3}$$

It is now straightforward to verify that an equilibrium in such an extended model can be determined by setting $\overline{w}_t = w_t$ (where w_t is given by [23]) and then obtaining the cohort-specific wages by setting $l_{t,s} = h_{t,s}$, and $L_t = (1-\phi)$ in equation (1.3) and solving for $w_{t,s}$. To see this, note that by making these substitutions and using (1.2) inside (1.3) leads to the per-worker income process

$$\frac{w_{t,s} h_{t,s}}{(1-\phi)} = \overline{w}_t q_{t,s} \ , \tag{1.4}$$

which coincides with the labor income process in the baseline model. Furthermore by setting $w_t = \overline{w}_t$, the market for total (composite) labor units clears by construction, whereas the cohort specific wages implied by (1.4) clear all cohort specific labor markets, since they satisfy equation (1.3) for all markets.

2 A multi-sector extension

It is straightforward to extend the model to allow for multiple sectors, with potentially different degrees of innovation within each sector. Such an extension can help illustrate that even when technological progress is different accross industries, the value premium is likely to be particularly salient *within* industries, as it is in the data.

To introduce such an extension, we modify the baseline setup, so that the production of the final good is given by

$$Y_t = Z_t \left(L_t^F\right)^{1-(\alpha_1+\alpha_2)} \left(\int_0^{A_t} x_{j,t}^{\alpha_1} dj\right) \left(\int_0^{B_t} \widetilde{x}_{j,t}^{\alpha_2} dj\right), \tag{2.1}$$

where $\alpha_1 > 0, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$, and $x_{j,t}$ denotes the intermediate input j in sector A and $\widetilde{x}_{j,t}$ denotes the intermediate input j in sector B. (To simplify the exposition and avoid inessential notation, specification (2.1) implicitly sets the weights $\omega_{j,t}$ on the intermediate goods equal to one). A_t and Z_t evolve as in the baseline version of the model and B_t evolves similarly to A_t, i.e.,

$$\log B_{t+1} = \log B_t + \widetilde{u}_{t+1},$$

where $\widetilde{u}_{t+1}$ is a non-negative random variable that captures technological advancements in sector B. We allow the shocks u_t and $\widetilde{u}_t$ to be correlated. At each point in time t, the representative final-good firm chooses L_t^F, $x_{j,t}$, and $\widetilde{x}_{j,t}$ so as to maximize its profits

$$\pi_t^F = \max_{L_t^F, x_{j,t}, \widetilde{x}_{j,t}} \left\{ Y_t - \int_0^{A_t} p_{j,t} x_{j,t} dj - \int_0^{B_t} \widetilde{p}_{j,t} \widetilde{x}_{j,t} dj - w_t L_t^F \right\}, \tag{2.2}$$

where $p_{j,t}$ and $\widetilde{p}_{j,t}$ are the prices of intermediate goods in sectors A and B, respectively, and w_t is the prevailing wage (per efficiency unit of labor).

Production of intermediate goods (in either sector) still takes the simple form described in the paper (i.e., it takes one unit of labor to produce one unit of intermediate good j, irrespective of the sector). Accordingly, the total labor demand of both sectors is

$$L_t^I = \int_0^{A_t} x_{j,t} dj + \int_0^{B_t} \widetilde{x}_{j,t} dj. \tag{2.3}$$

Finally, to simplify matters, we assume that new firms are specific to sectors and can own only sector-A or sector-B blueprints (but not both). Differentiating (2.2) leads to the following two first-order conditions with respect to $x_{j,t}$ and $\widetilde{x}_{j,t}$.

$$x_{j,t} = \left[\frac{p_{j,t}}{\alpha_1 Z_t \left(L_t^F\right)^{1-(\alpha_1+\alpha_2)} \left(\int_0^{B_t} \widetilde{x}_{j,t}^{\alpha_2} dj\right)}\right]^{\frac{1}{\alpha_1-1}}, \tag{2.4}$$

$$\widetilde{x}_{j,t} = \left[\frac{\widetilde{p}_{j,t}}{\alpha_2 Z_t \left(L_t^F\right)^{1-(\alpha_1+\alpha_2)} \left(\int_0^{A_t} x_{j,t}^{\alpha_1} dj\right)}\right]^{\frac{1}{\alpha_2-1}}. \tag{2.5}$$

Maximizing the profits of intermediate-good firms leads to the same first-order condition as in the baseline version of the model, namely:

$$p_{j,t} = \frac{w_t}{\alpha_1}, \tag{2.6}$$

$$\widetilde{p}_{j,t} = \frac{w_t}{\alpha_2}. \tag{2.7}$$

Combining (2.4) with (2.6), (2.5) with (2.7) and using the definition of Y_t leads to

$$x_{j,t} = \left[\frac{w_t}{\alpha_1^2 Y_t} \left(\int_0^{A_t} x_{j,t}^{\alpha_1} dj\right)\right]^{\frac{1}{\alpha_1-1}} \tag{2.8}$$

$$\widetilde{x}_{j,t} = \left[\frac{w_t}{\alpha_2^2 Y_t} \left(\int_0^{B_t} \widetilde{x}_{j,t}^{\alpha_2} dj\right)\right]^{\frac{1}{\alpha_2-1}} \tag{2.9}$$

Since all intermediate goods within a sector face the same demand curve and the same cost structure, we look for a symmetric equilibrium, in which $x_{j,t} = x_t$ and $\widetilde{x}_{j,t} = \widetilde{x}_t$. Under this supposition, $\int_0^{A_t} x_{j,t}^{\alpha_1} dj = A_t x_t^{\alpha_1}$ and $\int_0^{B_t} \widetilde{x}_{j,t}^{\alpha_2} dj = B_t \widetilde{x}_t^{\alpha_2}$, so that equations (2.8) and (2.9) simplify to

$$x_t = \alpha_1^2 \left(\frac{Y_t}{w_t}\right) \frac{1}{A_t}, \tag{2.10}$$

$$\widetilde{x}_t = \alpha_2^2 \left(\frac{Y_t}{w_t}\right) \frac{1}{B_t}. \tag{2.11}$$

Finally, the final-good firm's first-order condition with respect to labor gives $(1-\alpha_1-\alpha_2)\, Y_t = w_t L_t^F$, which implies that

$$\frac{Y_t}{w_t} = \frac{L_t^F}{1-\alpha_1-\alpha_2}. \tag{2.12}$$

Labor-market clearing can be expressed as

$$L_t^F + A_t x_t + B_t \widetilde{x}_t = (1-\phi) . \tag{2.13}$$

Using (2.12) inside (2.10) and (2.11), and then using the resulting expressions inside (2.13) and solving for L_t^F leads to

$$L_t^F = \frac{1-\alpha_1-\alpha_2}{1-\alpha_1-\alpha_2+\alpha_1^2+\alpha_2^2} (1-\phi) . \tag{2.14}$$

Combining (2.10), (2.11), (2.12), and (2.14) gives

$$x_t = \frac{\alpha_1^2}{1-\alpha_1-\alpha_2+\alpha_1^2+\alpha_2^2} \frac{1-\phi}{A_t}, \tag{2.15}$$

$$\widetilde{x}_t = \frac{\alpha_2^2}{1-\alpha_1-\alpha_2+\alpha_1^2+\alpha_2^2} \frac{1-\phi}{B_t}. \tag{2.16}$$

Combining (2.14) with (2.15) and (2.16) yields

$$Y_t = \frac{(1-\phi)(1-\alpha_1-\alpha_2)^{1-(\alpha_1+\alpha_2)} \alpha_1^{2\alpha_1} \alpha_2^{2\alpha_2}}{1-\alpha_1-\alpha_2+\alpha_1^2+\alpha_2^2} Z_t A_t^{1-\alpha_1} B_t^{1-\alpha_2}. \tag{2.17}$$

Equation (2.17) states that output is proportional to $Z_t A_t^{1-\alpha_1} B_t^{1-\alpha_2}$. From a practical perspective, this implies that the model with multiple sectors behaves like a single-sector model, where the technology shock u_t is replaced by a weighted sum of the technology shocks to the two sectors.[1] In particular all the conclusions regarding the displacement effect are unaltered, with the understanding that the shock u_t in the baseline model is now an appropriate weighted average of the shocks in the two sectors.

Even though the extension to multiple sectors adds little in terms of the model's general-equilibrium properties, it helps clarify that even when technological progress is concentrated in one sector, most of the cross-sectional differences in returns manifest themselves within

[1]To see this, note that

$$\Delta \log Y_t = \varepsilon_t + (1-\alpha_1) u_t + (1-\alpha_2) \widetilde{u}_t.$$

Defining $u_t^* = (1-\alpha_1) u_t + (1-\alpha_2) \widetilde{u}_t$ shows that the output growth in the multisector model is identical to the one in the single-sector model, with u_t^* defined appropriately to capture the total effect of all displacement shocks.

a sector, rather than across sectors. To see this, note first that per-firm profits in sector A and B are given by

$$\begin{aligned}\pi_t^A &= \alpha_1 (1-\alpha_1)\left(\frac{Y_t}{A_t}\right),\\ \pi_t^B &= \alpha_2 (1-\alpha_2)\left(\frac{Y_t}{B_t}\right).\end{aligned}$$

Now suppose that technological advancements are concentrated in one sector (say, sector A), so that u_t is random, but $\widetilde{u}_t$ is a constant. Take any stock in sector B. Since $\widetilde{u}_t$ is non-random, there will be no difference between the rates of return of different firms in sector B.[2] By contrast, stocks in sector A exhibit a non-trivial value premium, with "pure" growth options in sector A having a lower expected return than sector-B stocks (since they act as a hedge against the u-shock) and pure value stocks in sector A having higher expected returns than sector-B stocks. As a result, the "HML" factor in this economy is driven exclusively by return differentials *within* sector A.

3 More general endowment processes

We simplify some aspects of the model for tractability. One of the stylized assumptions is that innovating agents receive their blueprints "at birth." In reality, it takes time to start a new firm, and each cohort of agents does not innovate simultaneously. Moreover, innovation shocks u_t are more likely to follow a moving-average process rather than being independent, as we assume. We provide a simple example to illustrate why such frictions and perturbations

[2]To see this, let R_t^a denote the return of a "pure" asset in place in sector B, and R_t^o the return on a "pure" growth option in sector B. The definitions of R_t^a and R_t^o in the paper imply that $\log R_{t+1}^a - \log R_{t+1}^o$ is a non-random constant. Indeed, it is zero, since

$$\begin{aligned}1 &= E_t\left(e^{\log R_{t+1}^a}\frac{\xi_{t+1}}{\xi_t}\right) = E_t\left(e^{\log R_{t+1}^a-\log R_{t+1}^o}\times e^{\log R_{t+1}^o}\frac{\xi_{t+1}}{\xi_t}\right)\\ &= e^{\log R_{t+1}^a-\log R_{t+1}^o}\times E_t\left(e^{\log R_{t+1}^o}\frac{\xi_{t+1}}{\xi_t}\right) = e^{\log R_{t+1}^a-\log R_{t+1}^o}\end{aligned}$$

and hence $\log R_{t+1}^a = \log R_{t+1}^o$.

of the baseline model are unlikely to affect our conclusions about the long-run properties of the model-implied asset returns.

Suppose that all agents are born as workers with an initial endowment of labor efficiency units of $\overline{h}\,(1-\phi)\,q_{s,s}$. Furthermore, suppose that a fraction ϕ of them become entrepreneurs in the second period of their lives and permanently drop out of the workforce, whereas the ones that remain workers have an endowment of labor efficiency units equal to the baseline model from the second period of their lives onward, namely $\overline{h}(1+\delta)^{t-s}q_{t,s}$ for all $t \geq s+1$.[3] Finally, assume that agents can only access financial markets in the second period of their lives, while in the first period they consume their wage income. These assumptions capture the idea that an agent's "birth" cohort and the date at which that agent innovates may not coincide. Moreover, exclusion from markets captures in a stylized manner the idea that the agent cannot smooth consumption between the "birth" date and the innovation arrival date.

Repeating the argument of Section 3.2, the equilibrium stochastic discount factor in this modified setup is

$$\frac{\xi_{t+1}}{\xi_t} = \beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-1+\psi(1-\gamma)} \hat{v}(u_{t+1},u_t)^{-\gamma},$$

where

$$\hat{v}(u_{t+1},u_t) = (1-\lambda)^{-1}\left(1-\frac{\lambda y_t}{C_t}\right)^{-1}\left(1-\lambda\,(1-\lambda)\sum_{i\in\{w,e\}}\phi^i\frac{c^i_{t+1,t}}{C_{t+1}} - \lambda\frac{y_{t+1}}{C_{t+1}}\right)$$

and y_t denotes an agent's initial wage income. Furthermore, the same reasoning as in the proof of Lemma 4 implies that the variance of the permanent component of log consumption cohort effects equals $Var\left(\hat{v}(u_{t+1},u_t)\right)$.

This simple example illustrates the fact that the frictions affecting agents' life-cycle of earnings change the transitory dynamics of cohort effects, returns, and the stochastic discount factor. Such frictions do not alter our main qualitative conclusion that the permanent component of cohort effects captures the permanent component of the displacement factor, as reflected in the stochastic discount factor.

[3]Note that since agents are born with $\overline{h}\,(1-\phi)\,q_{s,s}$ rather than $\overline{h}q_{s,s}$ efficiency units, the total supply of labor efficiency units remains equal to the baseline model.

	Data	$\psi = 1$	$\kappa = 0.7$ baseline	$\kappa = 0.7$ modified
Aggregate (log) Consumption Growth rate	0.017	0.017	0.017	0.017
Aggregate (log) Consumption Volatility	0.033	0.032	0.032	0.033
Riskless Rate	0.010	0.057	0.025	0.017
Equity premium	0.061	0.046	0.026	0.033
Aggregate Earnings / Price	0.075	0.144	0.091	0.110
Dividend Volatility	0.112	0.107	0.078	0.104
Correl. (divid. growth, cons.growth)	0.2	0.2	0.293	0.155
Std $(\Delta\alpha_s^{\text{perm}})$	0.023	0.028	0.023	0.024
$\frac{cov\left(\Delta\alpha_s^{\text{perm}}, \log R^g - \log R^a\right)}{var\left(\Delta\alpha_s^{\text{perm}}\right)}$	3.92	3.866	5.370	6.766
Std $(\Delta w_s^{\text{perm}})$	0.022	0.023	0.023	0.024
Earnings / Price 90th Perc.	0.120	0.153	0.110	0.140
Earnings / Price 10th Perc.	0.04	0.065	0.034	0.044
Average Value premium	0.081	0.068	0.067	0.079
Std (Value Premium)	0.120	0.111	0.122	0.161
$\mathrm{E}\left(\log R^g - \log R^a\right)$		0.118	0.096	0.114

Table 1: Robustness Checks. The columns titled $\psi = 1$ and $\kappa = 0.7$ display results when the parameters ψ and κ are set equal to 1 and 0.7 respectively, while the rest of the parameters are kept at their baseline values ($\gamma = 10$). The last column displays results assuming that $\kappa = 0.7$, $\chi = 5$, $\nu = 0.06$, $\rho = 0.8$, $\omega = 0.87$ and the rest of the parameters are kept at their baseline values.

4 Robustness checks with respect to the parameters in the baseline model

Table 1 reports results from some simple robustness exercises. The column titled $\psi = 1$ helps isolate the effect of habit formation. This column shows how results change in the case where agents have standard constant relative risk aversion preferences ($\gamma = 10$). Comparing this table to Table 6 in the paper, it is apparent that the absence of habit formation increases slightly both the equity and the value premium. However, this comes at the cost of also increasing the riskless rate[4] and as a result all the earnings-to-price ratios. The next column ($\kappa = 0.7$) reduces κ to 0.7 while keeping the rest of the parameters unchanged. Recall that κ reflects the fraction of new blueprints accruing to new firms owned by arriving agents, while $(1 - \kappa)$ accrues to existing agents. In the baseline case scenario we choose $\kappa = 0.9$. We consider this a plausible value for the following reason: In a fully specified endogenous-innovation model with capital, where factors of production in the innovation sector are compensated for their marginal product, κ would capture the share of human capital, labor and skill in the innovation process (as opposed to the share of capital operated by pre-existing firms). Assuming that education, entrepreneurial skill and human effort are the most important scarce factors in the innovation process, one would expect κ to be close to 1. (For instance, in the seminal Romer model, labor is the exclusive factor of production in the development of new ideas). However, to examine the robustness of the results to this assumption, we also examine what happens when we choose these income shares to be similar to aggregate income shares in NIPA data. To that end we choose $\kappa = 0.7$. The next to last column reports results when all other parameters are kept at their base values, while the last column reports what happens when the rest of the parameters are also modified in order to match the volatility and the correlation of dividends. As can be seen, even though the results are slightly weaker when $\kappa = 0.7$, the model retains its power to explain a large fraction of the observed moments in the data.

[4]Inspection of equation (31) helps with the intuition behind this result: As ψ increases from 0 to 1, the exponent of $\frac{Y_{t+1}}{Y_t}$ decreases from -1 to $-\gamma$, resulting in a higher volatility of the pricing kernel, but also a stronger negative drift of the (log) stochastic discount factor.