10 Appendix

10.1 Expected values

We often generate one series (price, consumption, inflation) as an expected discounted sum of another (dividends, income, policy disturbances)

$$\begin{aligned} y_t &= E_t \sum_{j=0}^{\infty} \theta^j x_{t+j} \\ x_t &= b(L)\varepsilon_t \end{aligned}$$

Task 1 Find a representation for $y_t = a(L)\varepsilon_t$. The answer is (Hansen and Sargent (1980))

$$y_t = \left(\frac{Lb(L) - \theta b(\theta)}{L - \theta} \right) \varepsilon_t.$$

Here's why. Start by writing out

$$y_t^* = \sum_{j=0}^{\infty} \theta^j x_{t+j} = \frac{1}{1 - \theta L^{-1}} x_t = \frac{1}{1 - \theta L^{-1}} b(L)\varepsilon_t.$$

$$y_t^* = \begin{array}{ccccccc} & & & b_0\varepsilon_t & +b_1\varepsilon_{t-1} & +b_2\varepsilon_{t-2} & +... \\ & & +(\theta b_0\varepsilon_{t+1}) & +\theta b_1\varepsilon_t & +\theta b_2\varepsilon_{t-1} & +\theta b_3\varepsilon_{t-2} & +... \\ & +(\theta^2 b_0\varepsilon_{t+2}) & +(\theta^2 b_1\varepsilon_{t+1}) & +\theta^2 b_2\varepsilon_t & +\theta^2 b_3\varepsilon_{t-1} & +\theta^2 b_4\varepsilon_{t-2} & +... \\ +(\theta^3 b_0\varepsilon_{t+3}) & +(\theta^3 b_1\varepsilon_{t+2}) & +(\theta^3 b_2\varepsilon_{t+1}) & +\theta^3 b_3\varepsilon_t & +\theta^3 b_4\varepsilon_{t-1} & +\theta^3 b_5\varepsilon_{t-2} & +... \end{array}$$

Now, y_t is formed by simply getting rid of all the terms involving future ε_{t+j}, which I put in parentheses. Next sum the columns. For example, the ε_{t+1} term is

$$\theta b_0 + \theta^2 b_1 + \theta^3 b_2 + ... = \theta b(\theta)$$

Thus, we can write

$$\begin{aligned} y_t &= \left\{ \frac{b(L)}{1 - \theta L^{-1}} - \left[\theta b(\theta) L^{-1} + \theta^2 b(\theta) L^{-2} + \theta^3 b(\theta) L^{-3} + ... \right] \right\} \varepsilon_t \\ &= \left\{ \frac{b(L)}{1 - \theta L^{-1}} - b(\theta) \left[\theta L^{-1} + \theta^2 L^{-2} + \theta^3 L^{-3} + ... \right] \right\} \varepsilon_t \\ &= \left\{ \frac{b(L)}{1 - \theta L^{-1}} - \frac{b(\theta)\theta L^{-1}}{1 - \theta L^{-1}} \right\} \varepsilon_t \\ &= \left\{ \frac{Lb(L) - b(\theta)\theta}{L - \theta} \right\} \varepsilon_t \end{aligned}$$

Example. Suppose

$$x_t = \rho x_{t-1} + \varepsilon_t.$$

It's easy to work out by hand that

$$E_t \sum_{j=0} \theta^j x_{t+j} = \sum \theta^j \rho^j x_t = \frac{1}{1-\rho\theta} x_t = \frac{1}{1-\rho\theta} \frac{1}{1-\rho L} \varepsilon_t.$$

Our formula gives

$$\begin{aligned}
E_t \sum_{j=0} \theta^j x_{t+j} &= \left\{ \frac{\frac{L}{1-\rho L} - \frac{\theta}{1-\rho\theta}}{L-\theta} \right\} \varepsilon_t \\
&= \left\{ \frac{\frac{L(1-\rho\theta)-\theta(1-\rho L)}{(1-\rho L)(1-\rho\theta)}}{L-\theta} \right\} \varepsilon_t \\
&= \left\{ \frac{\frac{L-\theta}{(1-\rho L)(1-\rho\theta)}}{L-\theta} \right\} \varepsilon_t \\
&= \frac{1}{(1-\rho L)(1-\rho\theta)} \varepsilon_t
\end{aligned}$$

just as it should.

Task 2, reverse engineering Suppose you have a representation for $y_t = a(L)\varepsilon_t$. Construct an $x_t = b(L)\varepsilon_t$ that justifies it by $y_t = E_t \sum_{j=0}^{\infty} \theta^j x_{t+j}$. We want

$$a(L) = \frac{Lb(L) - \theta b(\theta)}{L-\theta}.$$

Solving,

$$a(L)(L-\theta) = Lb(L) - \theta b(\theta).$$

Evaluate at $L = 0$ to find $b(\theta)$

$$\begin{aligned}
a(0)(-\theta) &= -b(\theta)\theta \\
a(0) &= b(\theta)
\end{aligned}$$

Then substitute

$$\begin{aligned}
a(L)(L-\theta) &= Lb(L) - a(0)\theta \\
b(L) &= \frac{a(L)(L-\theta) + a(0)\theta}{L} \\
b(L) &= a(L)(1-\theta L^{-1}) + a(0)\theta L^{-1} \\
b(L) &= a(L) - \theta L^{-1}(a(L) - a(0))
\end{aligned}$$

That's the answer.

We can also write the answer out explicitly:

$$\begin{aligned}
b(L) &= a_0 + a_1 L + a_2 L^2 + a_3 L^3 + ... - \theta L^{-1}\left(a_1 L + a_2 L^2 + ...\right) \\
&= (a_0 - \theta a_1) + (a_1 - \theta a_2) L + (a_2 - \theta a_3) L^2 + ...
\end{aligned}$$

i.e.

$$b_j = a_j - \theta a_{j+1} \tag{54}$$

We can check,

$$\begin{aligned}
E_t \sum_{j=0}^{\infty} \theta^j x_{t+j} &= E_t \sum_{j=0}^{\infty} \theta^j b(L) \varepsilon_{t+j} \\
&= E_t \sum_{j=0}^{\infty} \theta^j \sum_{k=0}^{\infty} (a_k - \theta a_{k+1}) \varepsilon_{t+j-k}
\end{aligned}$$

$$\begin{aligned}
= \quad & (a_0 - \theta a_1) \varepsilon_t + (a_1 - \theta a_2) \varepsilon_{t-1} + (a_2 - \theta a_3) \varepsilon_{t-2} + ... \\
& + \theta \left[(a_1 - \theta a_2) \varepsilon_t + (a_2 - \theta a_3) \varepsilon_{t-1} + (a_3 - \theta a_4) \varepsilon_{t-2} + ... \right] \\
& + \theta^2 \left[(a_2 - \theta a_3) \varepsilon_t + (a_3 - \theta a_4) \varepsilon_{t-1} + (a_4 - \theta a_5) \varepsilon_{t-2} + ... \right]
\end{aligned}$$

$$= a_0 \varepsilon_t + a_1 \varepsilon_{t-1} + a_2 \varepsilon_{t-2} + ...$$

Our application. We have $y_t = -\phi^{-1} E_t \sum_{j=0}^{\infty} \phi^{-j} x_{t+j}$, i.e. $\theta = \phi^{-1}$ and we need to multiply x_t by an additional $-\phi$ after we're done. Equation (54) becomes

$$\begin{aligned}
b(L) &= -\phi \left[a(L) - \phi^{-1} L^{-1} (a(L) - a(0)) \right] \\
b(L) &= -\phi a(L) + L^{-1} (a(L) - a(0)) \\
b(L) &= \left(L^{-1} - \phi \right) a(L) - L^{-1} a(0)
\end{aligned}$$

10.2 The three-equation model

The standard three equation model is, in deviation form

$$\begin{aligned}
y_t &= E_t y_{t+1} - \sigma r_t \\
i_t &= r_t + E_t \pi_{t+1} \\
\pi_t &= \beta E_t \pi_{t+1} + \gamma y_t \\
i_t &= \phi_{\pi,0} \pi_t + \phi_{\pi,1} E_t \pi_{t+1} + \phi_{y,0} y_t + \phi_{y,1} E_t y_{t+1}
\end{aligned}$$

1. Express in standard form.

Eliminating i and r,

$$y_t = E_t y_{t+1} - \sigma \left(\phi_{\pi,0} \pi_t + \left(\phi_{\pi,1} - 1 \right) E_t \pi_{t+1} + \phi_{y,0} y_t + \phi_{y,1} E_t y_{t+1} \right).$$

$$\begin{aligned}
\left(1 - \sigma \phi_{y,1}\right) E_t y_{t+1} - \sigma \left(\phi_{\pi,1} - 1 \right) E_t \pi_{t+1} &= \left(1 + \sigma \phi_{y,0} \right) y_t + \sigma \phi_{\pi,0} \pi_t \\
\beta E_t \pi_{t+1} &= -\gamma y_t + \pi_t
\end{aligned}$$

$$\left[\begin{array}{ll} 1-\sigma\phi_{y,1} & -\sigma\left(\phi_{\pi,1}-1\right) \\ 0 & \beta \end{array}\right]\left[\begin{array}{c} E_t y_{t+1} \\ E_t \pi_{t+1} \end{array}\right]=\left[\begin{array}{ll} 1+\sigma\phi_{y,0} & \sigma\phi_{\pi,0} \\ -\gamma & 1 \end{array}\right]\left[\begin{array}{c} y_t \\ \pi_t \end{array}\right]$$

$$\left[\begin{array}{c} E_t y_{t+1} \\ E_t \pi_{t+1} \end{array}\right]=\left[\begin{array}{cc} \frac{1}{1-\sigma\phi_{y,1}} & \frac{\sigma\left(\phi_{\pi,1}-1\right)}{\beta\left(1-\sigma\phi_{y,1}\right)} \\ 0 & \frac{1}{\beta} \end{array}\right]\left[\begin{array}{ll} 1+\sigma\phi_{y,0} & \sigma\phi_{\pi,0} \\ -\gamma & 1 \end{array}\right]\left[\begin{array}{c} y_t \\ \pi_t \end{array}\right]$$

$$\left[\begin{array}{c} E_t y_{t+1} \\ E_t \pi_{t+1} \end{array}\right]=\left[\begin{array}{cc} \frac{1+\sigma\phi_{y,0}-\sigma\gamma\left(\phi_{\pi,1}-1\right)/\beta}{1-\sigma\phi_{y,1}} & \sigma\frac{\phi_{\pi,0}+\left(\phi_{\pi,1}-1\right)/\beta}{1-\sigma\phi_{y,1}} \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} \end{array}\right]\left[\begin{array}{c} y_t \\ \pi_t \end{array}\right]$$

2. Eigenvalues

The eigenvalues of the transition matrix are

$$\left\|\begin{array}{ll} \frac{1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}}-\lambda & \sigma\frac{\phi_{\pi,0}-(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}} \\ -\gamma/\beta & 1/\beta-\lambda \end{array}\right\|=0$$

$$\begin{aligned} \left(\frac{1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}}-\lambda\right)(1/\beta-\lambda)+\sigma\gamma\left(\frac{\phi_{\pi,0}-(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}}\right)/\beta &= 0 \\ \left[1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta-\lambda\left(1-\sigma\phi_{y,1}\right)\right](1-\lambda\beta)+\sigma\gamma\left(\phi_{\pi,0}-(1-\phi_{\pi,1})/\beta\right) &= 0 \end{aligned}$$

$$\begin{aligned} 0 \;=\; & \beta\left(1-\sigma\phi_{y,1}\right)\lambda^2-\left[\left(1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta\right)\beta+\left(1-\sigma\phi_{y,1}\right)\right]\lambda \\ & +1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta+\sigma\gamma\left(\phi_{\pi,0}-(1-\phi_{\pi,1})/\beta\right) \end{aligned}$$

$$\beta\left(1-\sigma\phi_{y,1}\right)\lambda^2-\left[1+\beta+\sigma\gamma(1-\phi_{\pi,1})+\sigma\beta\phi_{y,0}-\sigma\phi_{y,1}\right]\lambda+1+\sigma\phi_{y,0}+\sigma\gamma\phi_{\pi,0}=0$$

If $\sigma\phi_{y,1}\neq 1$,

$$\begin{aligned} \lambda \;=\; & \frac{1}{2\beta\left(1-\sigma\phi_{y,1}\right)}\left\{1+\beta+\sigma\gamma\left(1-\phi_{\pi,1}\right)+\sigma\beta\phi_{y,0}-\sigma\phi_{y,1}\right. \\ & \left.\pm\sqrt{\left(1+\beta+\sigma\gamma\left(1-\phi_{\pi,1}\right)+\sigma\beta\phi_{y,0}-\sigma\phi_{y,1}\right)^2-4\beta\left(1-\sigma\phi_{y,1}\right)\left(1+\sigma\phi_{y,0}+\sigma\gamma\phi_{\pi,0}\right)}\right\} \end{aligned}$$

If $\sigma\phi_{y,1}=1$,

$$\beta\left(1-\sigma\phi_{y,1}\right)\lambda^2-\left[1+\beta+\sigma\gamma(1-\phi_{\pi,1})+\sigma\beta\phi_{y,0}-\sigma\phi_{y,1}\right]\lambda+1+\sigma\phi_{y,0}+\sigma\gamma\phi_{\pi,0}=0$$

$$-\left[\beta+\sigma\gamma(1-\phi_{\pi,1})+\sigma\beta\phi_{y,0}\right]\lambda+1+\sigma\phi_{y,0}+\sigma\gamma\phi_{\pi,0}=0$$

$$\lambda=\frac{1+\sigma\left(\phi_{y,0}+\gamma\phi_{\pi,0}\right)}{\beta+\sigma\left[\gamma(1-\phi_{\pi,1})+\beta\phi_{y,0}\right]}$$

3. Characterizing the region of local determinacy

To find the regions of determinacy, write

$$\begin{aligned}
\lambda &= \frac{1}{2a}\left(b \pm \sqrt{b^2 - 4ac}\right) \\
a &\equiv \beta\left(1 - \sigma\phi_{y,1}\right) \\
b &\equiv 1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right) + \sigma\beta\phi_{y,0} - \sigma\phi_{y,1} \\
c &\equiv 1 + \sigma\phi_{y,0} + \sigma\gamma\phi_{\pi,0}
\end{aligned}$$

The boundaries $\|\lambda\| = 1$ are as follows.

1) $\sigma\phi_{y,1} \neq 1$, real roots $b^2 - 4ac > 0$, $\lambda = 1$:

$$\left(\phi_{\pi,0} + \phi_{\pi,1} - 1\right) + \frac{1-\beta}{\gamma}\left(\phi_{y,1} + \phi_{y,0}\right) = 0$$

2) $\sigma\phi_{y,1} \neq 1$, real roots $b^2 - 4ac > 0$, $\lambda = -1$:

$$\left(1 + \phi_{\pi,0} - \phi_{\pi,1}\right) - \frac{1+\beta}{\gamma}\left(\phi_{y,1} - \phi_{y,0}\right) = -2\frac{(1+\beta)}{\sigma\gamma}$$

3) $\sigma\phi_{y,1} \neq 1$, Complex roots $b^2 - 4ac < 0$,

$$\gamma\phi_{\pi,0} + \phi_{y,0} + \beta\phi_{y,1} = \frac{\beta - 1}{\sigma}.$$

4) $\sigma\phi_{y,1} = 1$, $\lambda = 1$,

$$\phi_{\pi,0} + \phi_{\pi,1} + \frac{(1-\beta)}{\sigma\gamma}\left(1 + \sigma\phi_{y,0}\right) = 1$$

5) $\sigma\phi_{y,1} = 1$, $\lambda = -1$:

$$\phi_{\pi,0} - \phi_{\pi,1} + \frac{(1+\beta)}{\sigma\gamma}\left(1 + \sigma\phi_{y,0}\right) = -1$$

Special cases used in plots

1. only $\phi_{\pi,0}$

$$\lambda = \frac{1}{2\beta}\left((1+\beta+\sigma\gamma) \pm \sqrt{(1+\beta+\sigma\gamma)^2 - 4\beta\left(1 + \sigma\gamma\phi_{\pi,0}\right)}\right)$$

The condition for real roots is $(1+\beta+\sigma\gamma)^2 - 4\beta\left(1+\sigma\gamma\phi_{\pi,0}\right) > 0$. The $\|\lambda\| = 1$ regions are then

$$\phi_{\pi,0} = 1$$

$$\phi_{\pi,0} = -\left(1 + 2\frac{(1+\beta)}{\sigma\gamma}\right)$$

for complex roots, we have

$$\phi_{\pi,0} = \frac{\beta - 1}{\sigma\gamma}$$

2. $\phi_{\pi,0}, \phi_{\pi,1}$

$$\lambda = \frac{1}{2\beta}\left(1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right) \pm \sqrt{\left(1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right)\right)^2 - 4\beta\left(1 + \sigma\gamma\phi_{\pi,0}\right)}\right)$$

The boundaries $\|\lambda\| = 1$ are as follows.

1) Real roots, $\lambda = 1$

$$\phi_{\pi,0} + \phi_{\pi,1} = 1$$

2) Real roots, $\lambda = -1$:

$$\phi_{\pi,0} - \phi_{\pi,1} = -\left(1 + 2\frac{(1+\beta)}{\sigma\gamma}\right)$$

3) Complex roots

$$\phi_{\pi,0} = \frac{\beta - 1}{\sigma\gamma}.$$

In the case $\phi_{\pi,0} = 0$, we have

$$\|\lambda\|^2 = \frac{1}{4\beta^2}4\beta = \frac{1}{\beta} > 1$$

in the entire complex root region. (The complex root region in the plot with $\phi_{\pi,0} = 0$ has a very small band of real roots surrounding the plotted complex roots, and these decline quickly to one at the plotted boundary.)

Detailed algebra for determinacy regions:

1) $\sigma\phi_{y,1} \neq 1$, real roots, $\lambda = 1$.

$$\frac{1}{2a}\left(b \pm \sqrt{b^2 - 4ac}\right) = 1$$

$$\begin{aligned} b \pm \sqrt{b^2 - 4ac} &= 2a \\ b^2 - 4ac &= (2a - b)^2 \end{aligned}$$

$$\begin{aligned} 0 = {} & \left(1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right) + \sigma\beta\phi_{y,0} - \sigma\phi_{y,1}\right)^2 - 4\beta\left(1 - \sigma\phi_{y,1}\right)\left(1 + \sigma\phi_{y,0} + \sigma\gamma\phi_{\pi,0}\right) \\ & - \left(2\beta - 2\beta\sigma\phi_{y,1} - \left(1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right) + \sigma\beta\phi_{y,0} - \sigma\phi_{y,1}\right)\right)^2 \end{aligned}$$

$$\begin{aligned} 0 &= -4\sigma\beta\phi_{y,0} - 4\beta\sigma\phi_{y,1} + 4\sigma^2\beta\phi_{y,0}\phi_{y,1} - 4\beta\sigma\gamma\phi_{\pi,1} - 4\beta\sigma\gamma\phi_{\pi,0} \\ &\quad -4\beta\sigma^2\phi_{y,1}\gamma - 4\beta^2\sigma^2\phi_{y,1}\phi_{y,0} + 4\sigma\beta^2\phi_{y,0} + 4\beta\sigma\gamma \\ &\quad +4\beta^2\sigma\phi_{y,1} - 4\beta^2\sigma^2\phi_{y,1}^2 + 4\beta\sigma^2\phi_{y,1}^2 + 4\beta\sigma^2\phi_{y,1}\gamma\phi_{\pi,0} + 4\beta\sigma^2\phi_{y,1}\gamma\phi_{\pi,1} \end{aligned}$$

$$\begin{aligned} 0 &= -\phi_{y,0} - \phi_{y,1} + \sigma\phi_{y,0}\phi_{y,1} - \gamma\phi_{\pi,1} - \gamma\phi_{\pi,0} - \sigma\phi_{y,1}\gamma - \beta\sigma\phi_{y,1}\phi_{y,0} \\ &\quad +\beta\phi_{y,0} + \gamma + \beta\phi_{y,1} - \beta\sigma\phi_{y,1}^2 + \sigma\phi_{y,1}^2 + \sigma\phi_{y,1}\gamma\phi_{\pi,0} + \sigma\phi_{y,1}\gamma\phi_{\pi,1} \end{aligned}$$

$$\left(1 - \sigma\phi_{y,1}\right)\left(\beta\phi_{y,1} - \phi_{y,1} - \phi_{y,0} + \gamma + \beta\phi_{y,0} - \gamma\phi_{\pi,0} - \gamma\phi_{\pi,1}\right) = 0$$

$$\left(1 - \sigma\phi_{y,1}\right)\left((\beta - 1)\left(\phi_{y,1} + \phi_{y,0}\right) - \gamma\left(\phi_{\pi,0} + \phi_{\pi,1} - 1\right)\right) = 0$$

We have already assumed $\sigma\phi_{y,1} \neq 1$,so

$$\frac{(\beta - 1)}{\gamma}\left(\phi_{y,1} + \phi_{y,0}\right) - \left(\phi_{\pi,0} + \phi_{\pi,1} - 1\right) = 0$$

$$\left(\phi_{\pi,0} + \phi_{\pi,1} - 1\right) + \frac{1 - \beta}{\gamma}\left(\phi_{y,1} + \phi_{y,0}\right) = 0$$

This identifies parameters at which *one* eigenvalue is equal to one. We also have to check that the other one is greater than one.

2) $\sigma\phi_{y,1} \neq 1$, real roots, $\lambda = -1$

$$\frac{1}{2a}\left(b + \sqrt{b^2 - 4ac}\right) = -1$$

$$b^2 - 4ac = (2a + b)^2$$

$$\begin{aligned} 0 &= \left(1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right) + \sigma\beta\phi_{y,0} - \sigma\phi_{y,1}\right)^2 - 4\beta\left(1 - \sigma\phi_{y,1}\right)\left(1 + \sigma\phi_{y,0} + \sigma\gamma\phi_{\pi,0}\right) \\ &\quad - \left(2\beta - 2\beta\sigma\phi_{y,1} + \left(1 + \beta + \sigma\gamma\left(1 - \phi_{\pi,1}\right) + \sigma\beta\phi_{y,0} - \sigma\phi_{y,1}\right)\right)^2 \end{aligned}$$

:

$$\begin{aligned} 0 &= -4\sigma\beta\phi_{y,0} + 12\beta\sigma\phi_{y,1} - 8\beta^2 - 8\beta + 4\sigma^2\beta\phi_{y,0}\phi_{y,1} + 4\beta\sigma\gamma\phi_{\pi,1} - 4\beta\sigma\gamma\phi_{\pi,0} \\ &\quad +4\beta\sigma^2\phi_{y,1}\gamma + 4\beta^2\sigma^2\phi_{y,1}\phi_{y,0} - 4\sigma\beta^2\phi_{y,0} - 4\beta\sigma\gamma \\ &\quad +12\beta^2\sigma\phi_{y,1} - 4\beta^2\sigma^2\phi_{y,1}^2 - 4\beta\sigma^2\phi_{y,1}^2 + 4\beta\sigma^2\phi_{y,1}\gamma\phi_{\pi,0} - 4\beta\sigma^2\phi_{y,1}\gamma\phi_{\pi,1} \end{aligned}$$

$$\left(\sigma\phi_{y,1} - 1\right)\left(-\beta\sigma\phi_{y,1} - \sigma\phi_{y,1} + \sigma\beta\phi_{y,0} + \sigma\gamma + \sigma\phi_{y,0} + \sigma\gamma\phi_{\pi,0} - \sigma\gamma\phi_{\pi,1} + 2 + 2\beta\right) = 0$$

$$-\beta\sigma\phi_{y,1} - \sigma\phi_{y,1} + \sigma\beta\phi_{y,0} + \sigma\gamma + \sigma\phi_{y,0} + \sigma\gamma\phi_{\pi,0} - \sigma\gamma\phi_{\pi,1} + 2 + 2\beta = 0$$

$$(1 + \beta)\left(\phi_{y,1} - \phi_{y,0}\right) + \gamma\left(\phi_{\pi,1} - \phi_{\pi,0} - 1\right) = 2\frac{(1 + \beta)}{\sigma}$$

$$\left(\phi_{\pi,1}-\phi_{\pi,0}-1\right)+\frac{(1+\beta)}{\gamma}\left(\phi_{y,1}-\phi_{y,0}\right)=2\frac{(1+\beta)}{\sigma\gamma}$$
$$\left(1-\phi_{\pi,1}+\phi_{\pi,0}\right)-\frac{(1+\beta)}{\gamma}\left(\phi_{y,1}-\phi_{y,0}\right)=-2\frac{(1+\beta)}{\sigma\gamma}$$

3) $\sigma\phi_{y,1}\neq 1$, Complex roots,

$$\left\|\frac{1}{2a}\left(b\pm\sqrt{b^2-4ac}\right)\right\|=1$$

$$\begin{aligned}\left(b-i\sqrt{\|(b^2-4ac)\|}\right)\left(b+i\sqrt{\|(b^2-4ac)\|}\right) &= \|2a\| \\ \left(b^2+\left\|\left(b^2-4ac\right)\right\|\right) &= (2a)^2\end{aligned}$$

the roots are complex because $b^2-4ac<0$

$$\left(b^2-\left(b^2-4ac\right)\right)=(2a)^2$$
$$4ac=4a^2$$

$$\begin{aligned} c &= a \\ 1+\sigma\phi_{y,0}+\sigma\gamma\phi_{\pi,0} &= \beta\left(1-\sigma\phi_{y,1}\right) \\ \gamma\phi_{\pi,0}+\phi_{y,0}+\beta\phi_{y,1} &= \frac{\beta-1}{\sigma}.\end{aligned}$$

In the special case $\sigma\phi_{y,1}=1$, we have

$$\lambda=\frac{1+\sigma\left(\phi_{y,0}+\gamma\phi_{\pi,0}\right)}{\beta+\sigma\left[\gamma(1-\phi_{\pi,1})+\beta\phi_{y,0}\right]}$$

4) $\lambda=1$:

$$1+\sigma\left(\phi_{y,0}+\gamma\phi_{\pi,0}\right)=\beta+\sigma\left[\gamma(1-\phi_{\pi,1})+\beta\phi_{y,0}\right]$$

$$\begin{aligned}\frac{1-\beta}{\sigma} &= \gamma(1-\phi_{\pi,1})+\beta\phi_{y,0}-\left(\phi_{y,0}+\gamma\phi_{\pi,0}\right) \\ \frac{1-\beta}{\sigma} &= -\gamma\left(\phi_{\pi,1}+\phi_{\pi,0}-1\right)+(\beta-1)\,\phi_{y,0} \\ -\frac{1-\beta}{\sigma\gamma} &= \left(\phi_{\pi,0}+\phi_{\pi,1}-1\right)+\frac{(1-\beta)}{\gamma}\phi_{y,0} \\ \left(\phi_{\pi,0}+\phi_{\pi,1}-1\right) &= -\frac{(1-\beta)}{\sigma\gamma}\left(1+\sigma\phi_{y,0}\right) \\ \phi_{\pi,0}+\phi_{\pi,1}+\frac{(1-\beta)}{\sigma\gamma}\left(1+\sigma\phi_{y,0}\right) &= 1\end{aligned}$$

5) $\lambda = -1$:

$$1+\sigma\left(\phi_{y,0}+\gamma\phi_{\pi,0}\right) = -\beta-\sigma\left[\gamma(1-\phi_{\pi,1})+\beta\phi_{y,0}\right]$$

$$\left(1+\phi_{\pi,0}-\phi_{\pi,1}\right) = -\frac{(1+\beta)}{\sigma\gamma}\sigma\phi_{y,0}-\frac{1+\beta}{\sigma\gamma}$$

$$\begin{aligned}
\left(1+\phi_{\pi,0}-\phi_{\pi,1}\right) &= -\frac{(1+\beta)}{\sigma\gamma}\left(1+\sigma\phi_{y,0}\right) \\
\phi_{\pi,0}-\phi_{\pi,1}+\frac{(1+\beta)}{\sigma\gamma}\left(1+\sigma\phi_{y,0}\right) &= -1
\end{aligned}$$

10.3 Estimated coefficients.

This section derives Equations (20) and (21).

The system is

$$\begin{aligned}
y_t &= E_t y_{t+1} - \sigma r_t + x_{dt} \\
i_t &= r_t + E_t \pi_{t+1} \\
\pi_t &= \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t} \\
i_t &= \phi_\pi \pi_t + x_{it}
\end{aligned}$$

Eliminate i, r to express the model in standard form,

$$\begin{aligned}
E_t y_{t+1} &= y_t + \sigma r_t - x_{dt} \\
E_t \pi_{t+1} &= \phi_\pi \pi_t + x_{it} - r_t \\
\beta E_t \pi_{t+1} &= \pi_t - \gamma y_t - x_{\pi t}
\end{aligned}$$

$$\begin{aligned}
E_t y_{t+1} &= y_t + \sigma\left(E_t\pi_{t+1} - \phi_\pi \pi_t - x_{it}\right) - x_{dt} \\
E_t y_{t+1} &= y_t + \frac{\sigma}{\beta}\pi_t - \frac{\sigma}{\beta}\gamma y_t - \frac{\sigma}{\beta}x_{\pi t} - \sigma\phi_\pi\pi_t - \sigma x_{it} - x_{dt} \\
E_t y_{t+1} &= \left(1-\frac{\sigma\gamma}{\beta}\right) y_t + \sigma\left(\frac{1}{\beta}-\phi_\pi\right)\pi_t - \frac{\sigma}{\beta}x_{\pi t} - \sigma x_{it} - x_{dt}
\end{aligned}$$

Thus, we solve

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{dt+1} \\ x_{\pi t+1} \\ x_{it+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{\sigma\gamma}{\beta} & \frac{\sigma}{\beta}-\sigma\phi_\pi & -1 & -\frac{\sigma}{\beta} & -\sigma \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} & 0 & -1 & 0 \\ 0 & 0 & \rho_d & 0 & 0 \\ 0 & 0 & 0 & \rho_\pi & 0 \\ 0 & 0 & 0 & 0 & \rho_i \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{dt} \\ x_{\pi t} \\ x_{it} \end{bmatrix} + \begin{bmatrix} \delta_{yt+1} \\ \delta_{\pi t+1} \\ \varepsilon_{dt+1} \\ \varepsilon_{\pi t+1} \\ \varepsilon_{it+1} \end{bmatrix}$$

Taking eigenvalues and eigenvectors of the transition matrix, we can express the solution as

$$\begin{aligned} z_{dt} &= \rho_d z_{dt-1} + \varepsilon_{dt} \\ z_{\pi t} &= \rho_\pi z_{\pi t-1} + \varepsilon_{\pi t} \\ z_{it} &= \rho_i z_{it-1} + \varepsilon_{it} \end{aligned}$$

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1-\rho_d\beta & \sigma\left(1-(1+\rho_\pi)\beta+\phi_\pi\beta^2\right) & 1-\rho_i\beta \\ \gamma & \beta^2(1-\rho_\pi)+\sigma\gamma(1-\beta) & \gamma \end{bmatrix} \begin{bmatrix} z_{dt} \\ z_{\pi t} \\ z_{it} \end{bmatrix}$$

$$\begin{aligned} x_{dt} &= \left((1-\rho_d)(1-\rho_d\beta)+\sigma\gamma(\rho_d-\phi_\pi)\right) z_{dt} \\ x_{\pi t} &= \left((1-\rho_\pi)(1-\rho_\pi\beta)+\sigma\gamma(\rho_\pi-\phi_\pi)\right) \beta z_{\pi t} \\ x_{it} &= \left((1-\rho_i)(1-\rho_i\beta)+\sigma\gamma(\rho_i-\phi_\pi)\right) / \sigma z_{it} \end{aligned}$$

$$i_t = \phi_\pi \pi_t + x_{it}$$

What do you get if you regress i_t on π_t?

$$\hat{\phi}_\pi = \phi_\pi + cov(\pi_t, x_{it})/var(\pi_t)$$

Since π_t loads on the shock x_{it}, the covariance is not zero.

$$\begin{aligned} var(\pi_t) &= \gamma^2\sigma_{zd}^2 + \left[\beta^2(1-\rho_\pi)+\sigma\gamma(1-\beta)\right]^2 \sigma_{z\pi}^2 + \gamma^2\sigma_{zi}^2 \\ cov(\pi_t, x_{it}) &= \left((1-\rho_i)(1-\rho_i\beta)+\sigma\gamma(\rho_i-\phi_\pi)\right)(\gamma/\sigma)\sigma_{zi}^2 \end{aligned}$$

$$\frac{cov(\pi_t, x_{it})}{var(\pi_t)} = \frac{\left[(1-\rho_i)(1-\rho_i\beta)+\sigma\gamma(\rho_i-\phi_\pi)\right](\gamma/\sigma)\sigma_{zi}^2}{\gamma^2\sigma_{zd}^2 + \left[\beta^2(1-\rho_\pi)+\sigma\gamma(1-\beta)\right]^2 \sigma_{z\pi}^2 + \gamma^2\sigma_{zi}^2}$$

In the special case that the π and d shocks are zero, we have

$$\frac{cov(\pi_t, x_{it})}{var(\pi_t)} = \frac{(1-\rho_i)(1-\rho_i\beta)}{\sigma\gamma} + (\rho_i - \phi_\pi)$$

The ϕ_π cancel, so the answer is

$$\hat{\phi}_\pi = \frac{(1-\rho_i)(1-\rho_i\beta)}{\sigma\gamma} + \rho_i$$

To evaluate the expected-inflation rule, the system is now

$$\begin{aligned} y_t &= E_t y_{t+1} - \sigma r_t + x_{dt} \\ i_t &= r_t + E_t\pi_{t+1} \\ \pi_t &= \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t} \\ i_t &= \phi_\pi E_t \pi_{t+1} + x_{it} \end{aligned}$$

Eliminate i, r to express the model in standard form,

$$\begin{aligned} E_t y_{t+1} &= y_t + \sigma r_t - x_{dt} \\ (1-\phi_\pi) E_t \pi_{t+1} &= x_{it} - r_t \\ \beta E_t \pi_{t+1} &= \pi_t - \gamma y_t - x_{\pi t} \end{aligned}$$

$$\begin{aligned} E_t y_{t+1} &= y_t + \sigma(1-\phi_\pi) E_t \pi_{t+1} - \sigma x_{it} - x_{dt} \\ E_t y_{t+1} &= y_t + \frac{\sigma}{\beta}(1-\phi_\pi)(\pi_t - \gamma y_t - x_{\pi t}) - \sigma x_{it} - x_{dt} \\ E_t y_{t+1} &= \left[1 - \frac{\sigma\gamma}{\beta}(1-\phi_\pi)\right] y_t + \frac{\sigma}{\beta}(1-\phi_\pi)\pi_t - \frac{\sigma}{\beta}(1-\phi_\pi) x_{\pi t} - \sigma x_{it} - x_{dt} \end{aligned}$$

Thus, we solve

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{dt+1} \\ x_{\pi t+1} \\ x_{it+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\sigma\gamma}{\beta}(1-\phi_\pi) & \frac{\sigma}{\beta}(1-\phi_\pi) & -1 & -\frac{\sigma}{\beta}(1-\phi_\pi) & -\sigma \\ -\frac{\gamma}{\beta} & \frac{1}{\beta} & 0 & -1 & 0 \\ 0 & 0 & \rho_d & 0 & 0 \\ 0 & 0 & 0 & \rho_\pi & 0 \\ 0 & 0 & 0 & 0 & \rho_i \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{dt} \\ x_{\pi t} \\ x_{it} \end{bmatrix} + \begin{bmatrix} \delta_{yt+1} \\ \delta_{\pi t+1} \\ \varepsilon_{dt+1} \\ \varepsilon_{\pi t+1} \\ \varepsilon_{it+1} \end{bmatrix}.$$

Taking eigenvectors, the solution is

$$\begin{aligned} z_{dt} &= \rho_d z_{dt-1} + \varepsilon_{dt} \\ z_{\pi t} &= \rho_\pi z_{\pi t-1} + \varepsilon_{\pi t} \\ z_{it} &= \rho_i z_{it-1} + \varepsilon_{it} \end{aligned}$$

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1-\rho_d\beta & \sigma(1-\phi_\pi)[1-\beta(1+\rho_\pi)] & 1-\rho_i\beta \\ \gamma & \beta^2(1-\rho_\pi) + \sigma\gamma(1-\beta)(1-\phi_\pi) & \gamma \end{bmatrix} \begin{bmatrix} z_{dt} \\ z_{\pi t} \\ z_{it} \end{bmatrix}$$

$$\begin{aligned} x_{dt} &= [(1-\rho_d)(1-\rho_d\beta) + \sigma\gamma\rho_d(1-\phi_\pi)] z_{dt} \\ x_{\pi t} &= \beta[(1-\rho_\pi)(1-\rho_\pi\beta) + \sigma\gamma\rho_\pi(1-\phi_\pi)] z_{\pi t} \\ x_{it} &= \frac{1}{\sigma}[(1-\rho_i)(1-\rho_i\beta) + \sigma\gamma\rho_i(1-\phi_\pi)] z_{it} \end{aligned}$$

$$i_t = \phi_\pi E_t \pi_{t+1} + x_{it} \tag{55}$$

Now, we want to run a regression of i_t on $E_t\pi_{t+1}$. Again, I specialize to $z_d = z_\pi = 0$. Then,

$$\begin{aligned} \pi_t &= \gamma z_{\pi_t} \\ E_t \pi_{t+1} &= \gamma \rho_i z_{it} \end{aligned}$$

With two or fewer shocks, we can recover the shocks from the observable variables, so there is no issue that E_t formed by observable instruments gives less information than E_t

formed on the full information set, i.e. seeing the z. Thus, when we run regression (55), the result is

$$\begin{aligned}
\hat{\phi}_\pi &= \phi_\pi + \frac{cov\left(x_{it}, \gamma\rho_i z_{it}\right)}{var(\gamma\rho_i z_{it})} \\
&= \phi_\pi + \frac{1}{\sigma}\frac{\left[\left(1-\rho_i\right)\left(1-\rho_i\beta\right) + \sigma\gamma\rho_i\left(1-\phi_\pi\right)\right]\gamma\rho_i}{\gamma^2\rho_i^2} \\
\hat{\phi}_\pi &= 1 + \frac{\left(1-\rho_i\right)\left(1-\rho_i\beta\right)}{\sigma\gamma\rho_i}
\end{aligned}$$

10.4 Identification in the three-equation model

This section presents the algebra for Section 4.3. The characterization of the determinacy region is also used in "Inflation determination with Taylor rules."

1. Express the model in standard form

Eliminate i_t from (41)-(41), to produce a system with two endogenous variables y_t and π_t

$$y_t = E_t y_{t+1} - \sigma\left(\phi_{\pi,0}\pi_t + \left(\phi_{\pi,1}-1\right)E_t\pi_{t+1} + \phi_{y,0}y_t + \phi_{y,1}E_t y_{t+1} + x_{it} + \theta_y x_{yt} + \theta_\pi x_{\pi t}\right) + x_{yt}$$

$$(1-\sigma\phi_{y,1})E_t y_{t+1} + \sigma\left(1-\phi_{\pi,1}\right)E_t\pi_{t+1} = \left(1+\sigma\phi_{y,0}\right)y_t + \sigma\phi_{\pi,0}\pi_t + \sigma x_{it} + \left(\sigma\theta_y - 1\right)x_{yt} + \sigma\theta_\pi x_{\pi t}$$

Express the model in standard form,

$$\begin{bmatrix} 1-\sigma\phi_{y,1} & \sigma(1-\phi_{\pi,1}) & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{it+1} \\ x_{yt+1} \\ x_{\pi t+1} \end{bmatrix}$$
$$= \begin{bmatrix} 1+\sigma\phi_{y,0} & \sigma\phi_{\pi,0} & \sigma & \sigma\theta_y - 1 & \sigma\theta_\pi \\ -\gamma & 1 & 0 & 0 & 1 \\ 0 & 0 & \rho_i & 0 & 0 \\ 0 & 0 & 0 & \rho_y & 0 \\ 0 & 0 & 0 & 0 & \rho_\pi \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{it} \\ x_{yt} \\ x_{\pi t} \end{bmatrix} + \begin{bmatrix} \delta_{yt+1} \\ \delta_{\pi t+1} \\ \varepsilon_{it+1} \\ \varepsilon_{yt+1} \\ \varepsilon_{\pi t+1} \end{bmatrix}$$

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{it+1} \\ x_{yt+1} \\ x_{\pi t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\sigma\phi_{y,1}} & -\frac{\sigma(1-\phi_{\pi,1})}{\beta\left(1-\sigma\phi_{y,1}\right)} & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\sigma\phi_{y,0} & \sigma\phi_{\pi,0} & \sigma & \sigma\theta_y - 1 & \sigma\theta_\pi \\ -\gamma & 1 & 0 & 0 & 1 \\ 0 & 0 & \rho_i & 0 & 0 \\ 0 & 0 & 0 & \rho_y & 0 \\ 0 & 0 & 0 & 0 & \rho_\pi \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{it} \\ x_{yt} \\ x_{\pi t} \end{bmatrix} + \ldots$$

I ignore the errors, since the covariance matrix of the shocks has no observable implications.

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ x_{it+1} \\ x_{yt+1} \\ x_{\pi t+1} \end{bmatrix} = \begin{bmatrix} \frac{1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}} & \sigma\frac{\phi_{\pi,0}-(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}} & \frac{\sigma}{1-\sigma\phi_{y,1}} & \frac{\sigma\theta_y-1}{1-\sigma\phi_{y,1}} & \sigma\frac{\theta_\pi-(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}} \\ -\gamma/\beta & 1/\beta & 0 & 0 & 1/\beta \\ 0 & 0 & \rho_i & 0 & 0 \\ 0 & 0 & 0 & \rho_y & 0 \\ 0 & 0 & 0 & 0 & \rho_\pi \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ x_{it} \\ x_{yt} \\ x_{\pi t} \end{bmatrix} + ...$$

2. Eigenvalues of the transition matrix.

The eigenvalues of this transition matrix are ρ_i, ρ_y, ρ_π and the eigenvalues of the upper left-hand block,

$$\left\Vert \begin{matrix} \frac{1+\sigma\phi_{y,0}+\sigma\gamma(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}} - \lambda & \sigma\frac{\phi_{\pi,0}-(1-\phi_{\pi,1})/\beta}{1-\sigma\phi_{y,1}} \\ -\gamma/\beta & 1/\beta - \lambda \end{matrix} \right\Vert = 0$$

These are the same as analyzed in section 10.2.

4. Eigenvectors of the transition matrix.

I let Maple (in Scientific Workplace) find eigenvectors, using the compute, matrix, nullspace basis command. The eigenvectors of the stationary eigenvalues ρ_i, ρ_y, ρ_π are :

$$\begin{bmatrix} 1-\beta\rho_i \\ \gamma \\ -\frac{1}{\sigma}\alpha_i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-\rho_y\beta \\ \gamma \\ 0 \\ \frac{1}{1-\sigma\theta_y}\alpha_y \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma\left[(1-\beta\rho_\pi)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1}-1\right)\right] \\ (1-\rho_\pi) + \sigma\gamma\theta_\pi + \sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\ 0 \\ 0 \\ -\alpha_\pi \end{bmatrix}.$$

where

$$\begin{aligned} \alpha_i &\equiv (1-\rho_i)(1-\rho_i\beta) + \sigma\gamma\left(\phi_{\pi,0}+\rho_i\left(\phi_{\pi,1}-1\right)\right) + \sigma(1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right) \\ \alpha_y &\equiv \left(1-\rho_y\right)\left(1-\rho_y\beta\right) + \sigma\gamma\left(\phi_{\pi,0}+\rho_y\left(\phi_{\pi,1}-1\right)\right) + \sigma\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right) \\ \alpha_\pi &\equiv (1-\rho_\pi)(1-\rho_\pi\beta) + \sigma\gamma\left(\phi_{\pi,0}+\rho_\pi\left(\phi_{\pi,1}-1\right)\right) + \sigma(1-\rho_\pi\beta)\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \end{aligned}$$

If $\theta_y = 1/\sigma$, the second eigenvector collapses to

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We now have the model solution, i.e.

$$\begin{aligned} z_t &= \varrho z_{t-1} + \varepsilon_t \\ \varrho &= \begin{matrix} \rho_i & & \\ & \rho_y & \\ & & \rho_\pi \end{matrix} \end{aligned}$$

and

$$
\begin{bmatrix} i_t \\ y_t \\ x_{it} \\ x_{yt} \\ x_{\pi t} \end{bmatrix} = \begin{bmatrix} 1-\beta\rho_i & 1-\rho_y\beta & \sigma\left[(1-\beta\rho_\pi)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1}-1\right)\right] \\ \gamma & \gamma & (1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\ -\frac{1}{\sigma}\alpha_i & 0 & 0 \\ 0 & \frac{1}{1-\sigma\theta_y}\alpha_y & 0 \\ 0 & 0 & -\alpha_\pi \end{bmatrix} \begin{bmatrix} z_{it} \\ z_{yt} \\ z_{\pi t} \end{bmatrix}
$$

5. Put interest rates back in

We can now add the model's predictions for i_t which is also observable

$$
i_t = \phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_{y,0}y_t + \phi_{y,1}E_ty_{t+1} + x_{it} + \theta_\pi x_{\pi t} + \theta_y x_{yt}
$$

$$
\begin{aligned}
= \; & \phi_{\pi,0}\left(\gamma z_{it} + \gamma z_{yt} + \left((1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right)z_{\pi t}\right) \\
& +\phi_{\pi,1}\left(\left(\gamma\rho_i z_{it} + \gamma\rho_y z_{yt} + \left((1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right)\rho_\pi z_{\pi t}\right)\right) \\
& +\phi_{y,0}\left((1-\beta\rho_i)z_{it} + \left(1-\beta\rho_y\right)z_{yt} + \sigma\left[(1-\beta\rho_\pi)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1}-1\right)\right]z_{\pi t}\right) \\
& +\phi_{y,1}\left((1-\beta\rho_i)\rho_i z_{it} + \left(1-\beta\rho_y\right)\rho_y z_{yt} + \sigma\left[(1-\beta\rho_\pi)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1}-1\right)\right]\rho_\pi z_{\pi t}\right) \\
& +x_{it} + \phi_\pi x_{\pi t} + \phi_y x_{yt}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \left[\gamma\left(\phi_{\pi,0}+\rho_i\phi_{\pi,1}\right) + (1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right)\right]z_{it} + x_{it} \\
& +\left[\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right) + \left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)\right]z_{yt} + \theta_y x_{yt} \\
& +\left\{\left[(1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right]\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)\right. \\
& \left.+\sigma\left[(1-\beta\rho_\pi)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1}-1\right)\right]\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right\}z_{\pi t} + \theta_\pi x_{\pi t}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \left[\gamma\left(\phi_{\pi,0}+\rho_i\phi_{\pi,1}\right) + (1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right) - \frac{\alpha_i}{\sigma}\right]z_{it} \\
& +\left[\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right) + \left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right) + \frac{\alpha_y\theta_y}{1-\sigma\theta_y}\right]z_{yt} \\
& +\left\{\left[(1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right]\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)\right. \\
& \left.+\sigma\left[(1-\beta\rho_\pi)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1}-1\right)\right]\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) - \alpha_\pi\theta_\pi\right\}z_{\pi t}
\end{aligned}
$$

Simplifying the terms,

$$
\begin{aligned}
& \gamma\left(\phi_{\pi,0}+\rho_i\phi_{\pi,1}\right) + (1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right) - \frac{\alpha_i}{\sigma} \\
= \; & \gamma\left(\phi_{\pi,0}+\rho_i\phi_{\pi,1}\right) + (1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right) \\
& -\frac{1}{\sigma}\left((1-\rho_i)(1-\rho_i\beta) + \sigma\gamma\left(\phi_{\pi,0}+\rho_i\left(\phi_{\pi,1}-1\right)\right) + \sigma(1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right)\right) \\
= \; & \gamma\rho_i - \frac{1}{\sigma}(1-\rho_i)(1-\rho_i\beta) - (1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right) + (1-\beta\rho_i)\left(\phi_{y,0}+\rho_i\phi_{y,1}\right) \\
= \; & \gamma\rho_i - \frac{1}{\sigma}(1-\rho_i)(1-\rho_i\beta)
\end{aligned}
$$

$$
\begin{aligned}
& \gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\frac{\alpha_y\theta_y}{1-\sigma\theta_y} \\
= & \gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right) \\
& +\frac{\theta_y}{1-\sigma\theta_y}\left(\left(1-\rho_y\right)\left(1-\rho_y\beta\right)+\sigma\gamma\left(\phi_{\pi,0}+\rho_y\left(\phi_{\pi,1}-1\right)\right)+\sigma\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)\right) \\
& +\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right) \\
= & \gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\frac{\sigma\gamma\theta_y}{1-\sigma\theta_y}\left(\phi_{\pi,0}+\rho_y\left(\phi_{\pi,1}-1\right)\right) \\
& +\frac{\theta_y}{1-\sigma\theta_y}\left[\left(1-\rho_y\right)\left(1-\rho_y\beta\right)+\sigma\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)\right]+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right) \\
= & \frac{\gamma}{1-\sigma\theta_y}\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\frac{\theta_y}{1-\sigma\theta_y}\left(\left(1-\rho_y\right)\left(1-\rho_y\beta\right)-\sigma\gamma\rho_y\right) \\
& +\frac{\sigma\theta_y}{1-\sigma\theta_y}\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right) \\
= & \frac{\gamma}{1-\sigma\theta_y}\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\frac{\theta_y}{1-\sigma\theta_y}\left(\left(1-\rho_y\right)\left(1-\rho_y\beta\right)-\sigma\gamma\rho_y\right)+\frac{1}{1-\sigma\theta_y}\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right) \\
= & \frac{1}{1-\sigma\theta_y}\left\{\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta_y\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \left[(1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right]\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right) \\
& +\sigma\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right]\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)-\alpha_\pi\theta_\pi
\end{aligned}
$$

$$
\begin{aligned}
& \left((1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right)\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right) \\
& -(1-\rho_\pi)(1-\rho_\pi\beta)\theta_\pi-\sigma\gamma\left(\phi_{\pi,0}+\rho_\pi\left(\phi_{\pi,1}-1\right)\right)\theta_\pi-\sigma(1-\rho_\pi\beta)\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\theta_\pi \\
& +\sigma\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right]\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\
= & (1-\rho_\pi)\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)+\left[\sigma\gamma\rho_\pi-(1-\rho_\pi)(1-\rho_\pi\beta)\right]\theta_\pi \\
& +\sigma\left[\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)-(1-\rho_\pi\beta)\theta_\pi\right]\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\
& +\sigma\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right]\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\
= & (1-\rho_\pi)\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)+\left[\sigma\gamma\rho_\pi-(1-\rho_\pi)(1-\rho_\pi\beta)\right]\theta_\pi \\
& +\sigma\left\{\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)-(1-\rho_\pi\beta)\theta_\pi+(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right\}\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\
= & (1-\rho_\pi)\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)+\rho_\pi\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)+\left[\sigma\gamma\rho_\pi-(1-\rho_\pi)(1-\rho_\pi\beta)\right]\theta_\pi
\end{aligned}
$$

6. Transition matrix for all observables

Adding the i_t loadings on each z to the eigenvectors found so far, and ignoring the x shocks, the transition matrix for observables has the following expression

$$
\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{bmatrix} = Qz_t
$$

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{bmatrix} = Q \begin{bmatrix} \rho_i & & \\ & \rho_y & \\ & & \rho_\pi \end{bmatrix} Q^{-1} \begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} + \text{shocks}$$

$$\begin{aligned}
Q_{:,1} &= \begin{bmatrix} 1-\beta\rho_i \\ \gamma \\ \gamma\rho_i - \frac{1}{\sigma}(1-\rho_i)(1-\rho_i\beta) \end{bmatrix} \\
Q_{:,2} &= \begin{bmatrix} 1-\beta\rho_y \\ \gamma \\ \frac{1}{1-\sigma\theta_y}\left\{\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta_y\right\} \end{bmatrix} \\
Q_{:,3} &= \begin{bmatrix} \sigma\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right] \\ (1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right) \\ (1-\rho_\pi)\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)+\rho_\pi\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)+\left[\sigma\gamma\rho_\pi-(1-\rho_\pi)(1-\rho_\pi\beta)\right]\theta_\pi \end{bmatrix}
\end{aligned}$$

$Q_{:,1}$ and $Q_{1:2,2}$ identify β, γ, σ. The remaining Q are linear functions of the θ, ϕ parameters. Furthermore, $Q_{3,3}$ is redundant, $-\frac{1}{\sigma}(1-\rho_\pi)Q_{3,1}+\rho_\pi Q_{3,2}=Q_{3,3}$ Thus, all our information about the ϕ and θ parameters comes down to two restrictions,

$$\frac{1}{\gamma(1-\sigma\theta_y)}\left\{\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta_y\right\} = \frac{Q_{3,2}}{Q_{2,2}}$$

$$\frac{(1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)}{\sigma\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right]} = \frac{Q_{1,3}}{Q_{2,3}}$$

7. *Solving explicitly for ϕ, θ that are observationally equivalent to a given ϕ^*, θ^**

Q comes from ϕ^*, θ^* :

$$\begin{aligned}
& \frac{\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta_y}{\gamma\left(1-\sigma\theta_y\right)} \\
= & \frac{\gamma\left(\phi^*_{\pi,0}+\rho_y\phi^*_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi^*_{y,0}+\rho_y\phi^*_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta^*_y}{\gamma\left(1-\sigma\theta^*_y\right)}
\end{aligned}$$

$$\begin{aligned}
& \frac{(1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)}{\sigma\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right]} \\
= & \frac{(1-\rho_\pi)+\sigma\gamma\theta^*_\pi+\sigma\left(\phi^*_{y,0}+\rho_\pi\phi^*_{y,1}\right)}{\sigma\left[(1-\beta\rho_\pi)\theta^*_\pi-\phi^*_{\pi,0}-\rho_\pi\left(\phi^*_{\pi,1}-1\right)\right]}
\end{aligned}$$

Working on the first equation

$$\begin{aligned}
& \left(\gamma\left(\phi_{\pi,0}+\rho_y\phi_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi_{y,0}+\rho_y\phi_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta_y\right)\left(1-\sigma\theta^*_y\right) \\
= & \left(\gamma\left(\phi^*_{\pi,0}+\rho_y\phi^*_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\phi^*_{y,0}+\rho_y\phi^*_{y,1}\right)+\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y\right)\theta^*_y\right)\left(1-\sigma\theta_y\right)
\end{aligned}$$

define $\tilde{\phi} = \phi - \phi^*$, etc.,

$$\begin{aligned}
& \gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\tilde{\theta}_y \\
& -\left(\gamma\left(\phi_{\pi,0} + \rho_y \phi_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\phi_{y,0} + \rho_y \phi_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\theta_y\right)\left(\sigma\theta_y^*\right) \\
= \; & \left(\gamma\left(\phi^*_{\pi,0} + \rho_y \phi^*_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\phi^*_{y,0} + \rho_y \phi^*_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\theta^*_y\right)\left(-\sigma\theta_y\right)
\end{aligned}$$

$$\begin{aligned}
xy^* &= x^* y \\
\left(x - x^*\right)y^* + x^* y^* &= x^*\left(y - y^*\right) + x^* y^* \\
\left(x - x^*\right)y^* &= x^*\left(y - y^*\right)
\end{aligned}$$

$$\begin{aligned}
& \gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\tilde{\theta}_y \\
& -\left(\gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\tilde{\theta}_y\right)\sigma\theta_y^* \\
= \; & \left(\gamma\left(\phi^*_{\pi,0} + \rho_y \phi^*_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\phi^*_{y,0} + \rho_y \phi^*_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\theta^*_y\right)\left(-\sigma\tilde{\theta}_y\right)
\end{aligned}$$

$$\begin{aligned}
& \left(1 - \sigma\theta_y^*\right)\gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(1 - \sigma\theta_y^*\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right) \\
& + \left(1 - \sigma\theta_y^*\right)\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\tilde{\theta}_y \\
= \; & \left(\gamma\left(\phi^*_{\pi,0} + \rho_y \phi^*_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\phi^*_{y,0} + \rho_y \phi^*_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\theta^*_y\right)\left(-\sigma\tilde{\theta}_y\right)
\end{aligned}$$

$$\begin{aligned}
& \left(1 - \sigma\theta_y^*\right)\left\{\gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right)\right\} = \\
= \; & -\left\{\sigma\gamma\left(\phi^*_{\pi,0} + \rho_y \phi^*_{\pi,1}\right) + \sigma\left(1 - \beta\rho_y\right)\left(\phi^*_{y,0} + \rho_y \phi^*_{y,1}\right)\right. \\
& \left. + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\sigma\theta^*_y + \left(1 - \sigma\theta_y^*\right)\left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\right\}\tilde{\theta}_y
\end{aligned}$$

$$\begin{aligned}
0 = \; & \left(1 - \sigma\theta_y^*\right)\left\{\gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right)\right\} \\
& + \left\{\sigma\gamma\left(\phi^*_{\pi,0} + \rho_y \phi^*_{\pi,1}\right) + \sigma\left(1 - \beta\rho_y\right)\left(\phi^*_{y,0} + \rho_y \phi^*_{y,1}\right) + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\right\}\tilde{\theta}_y
\end{aligned}$$

$$\begin{aligned}
0 = \; & \left(1 - \sigma\theta_y^*\right)\left\{\gamma\left(\tilde{\phi}_{\pi,0} + \rho_y \tilde{\phi}_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\tilde{\phi}_{y,0} + \rho_y \tilde{\phi}_{y,1}\right)\right\} \\
& + \left\{\gamma\left(\phi^*_{\pi,0} + \rho_y \phi^*_{\pi,1}\right) + \left(1 - \beta\rho_y\right)\left(\phi^*_{y,0} + \rho_y \phi^*_{y,1}\right)\right\}\sigma\tilde{\theta}_y + \left(\left(1-\rho_y\right)\left(1-\beta\rho_y\right) - \sigma\gamma\rho_y\right)\tilde{\theta}_y
\end{aligned}$$

Working on the second equation

$$\frac{\left(1-\rho_\pi\right) + \sigma\gamma\theta_\pi + \sigma\left(\phi_{y,0} + \rho_\pi \phi_{y,1}\right)}{\sigma\left[\left(1-\beta\rho_\pi\right)\theta_\pi - \phi_{\pi,0} - \rho_\pi\left(\phi_{\pi,1} - 1\right)\right]} = \frac{\left(1-\rho_\pi\right) + \sigma\gamma\theta^*_\pi + \sigma\left(\phi^*_{y,0} + \rho_\pi \phi^*_{y,1}\right)}{\sigma\left[\left(1-\beta\rho_\pi\right)\theta^*_\pi - \phi^*_{\pi,0} - \rho_\pi\left(\phi^*_{\pi,1} - 1\right)\right]}$$

$$\begin{aligned} & \left((1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right)\left[(1-\beta\rho_\pi)\theta_\pi^*-\phi_{\pi,0}^*-\rho_\pi\left(\phi_{\pi,1}^*-1\right)\right] \\ = & \left((1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right)\left[(1-\beta\rho_\pi)\theta_\pi-\phi_{\pi,0}-\rho_\pi\left(\phi_{\pi,1}-1\right)\right] \end{aligned}$$

$$\begin{aligned} & \left((1-\rho_\pi)+\sigma\gamma\theta_\pi+\sigma\left(\phi_{y,0}+\rho_\pi\phi_{y,1}\right)\right)\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right] \\ = & \left((1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right)\left[(1-\beta\rho_\pi)\theta_\pi-\left(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}\right)+\rho_\pi\right] \end{aligned}$$

$$\begin{aligned} xy^* &= x^*y \\ (x-x^*)\,y^* &= x^*(y-y^*) \end{aligned}$$

$$\begin{aligned} & \left(\sigma\gamma\tilde{\theta}_\pi+\sigma\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right)\right)\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right] \\ = & \left((1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right)\left[(1-\beta\rho_\pi)\tilde{\theta}_\pi-\left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right)\right] \end{aligned}$$

$$0=\left\{\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right]\sigma\gamma-\left((1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right)(1-\beta\rho_\pi)\right\}\tilde{\theta}_\pi$$

$$\begin{aligned} & +\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right]\sigma\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right) \\ & +\left[(1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right]\left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right) \end{aligned}$$

$$\begin{aligned} 0 = & \left\{(1-\beta\rho_\pi)\sigma\gamma\theta_\pi^*-\sigma\gamma\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\sigma\gamma\rho_\pi-(1-\rho_\pi)(1-\beta\rho_\pi)\right. \\ & \left.-(1-\beta\rho_\pi)\sigma\gamma\theta_\pi^*-\sigma(1-\beta\rho_\pi)\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right\}\tilde{\theta}_\pi \end{aligned}$$

$$\begin{aligned} & +\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right]\sigma\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right) \\ & +\left((1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right)\left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right) \end{aligned}$$

$$\begin{aligned} 0 = & \left\{\begin{array}{c} +\sigma\gamma\rho_\pi-(1-\rho_\pi)(1-\beta\rho_\pi)-\sigma\gamma\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right) \\ -\sigma(1-\beta\rho_\pi)\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right) \end{array}\right\}\tilde{\theta}_\pi \\ & +\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right]\sigma\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right) \\ & +\left((1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right)\left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right) \end{aligned}$$

In sum, we have two linear equations in the 6 unknowns $\tilde{\theta}_\pi, \tilde{\phi}_y, \tilde{\phi}_{\pi,0}, \tilde{\phi}_{\pi,1}, \tilde{\phi}_{y,0}, \tilde{\phi}_{y,1}$

$$\begin{aligned} 0 = & \left\{\left(1-\rho_y\right)\left(1-\beta\rho_y\right)-\sigma\gamma\rho_y+\sigma\gamma\left(\phi_{\pi,0}^*+\rho_y\phi_{\pi,1}^*\right)+\sigma\left(1-\beta\rho_y\right)\left(\phi_{y,0}^*+\rho_y\phi_{y,1}^*\right)\right\}\tilde{\theta}_y \\ & +\left(1-\sigma\theta_y^*\right)\left\{\gamma\left(\tilde{\phi}_{\pi,0}+\rho_y\tilde{\phi}_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\tilde{\phi}_{y,0}+\rho_y\tilde{\phi}_{y,1}\right)\right\} \\ 0 = & -\left\{(1-\rho_\pi)(1-\beta\rho_\pi)-\sigma\gamma\rho_\pi+\sigma\gamma\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\sigma(1-\beta\rho_\pi)\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right\}\tilde{\theta}_\pi \\ & +\left[(1-\beta\rho_\pi)\theta_\pi^*-\left(\phi_{\pi,0}^*+\rho_\pi\phi_{\pi,1}^*\right)+\rho_\pi\right]\sigma\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right) \\ & +\left[(1-\rho_\pi)+\sigma\gamma\theta_\pi^*+\sigma\left(\phi_{y,0}^*+\rho_\pi\phi_{y,1}^*\right)\right]\left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right) \end{aligned}$$

8. Special cases

a) With $\theta = 0$:

$$
\begin{aligned}
0 &= \gamma\left(\tilde{\phi}_{\pi,0}+\rho_y\tilde{\phi}_{\pi,1}\right)+\left(1-\beta\rho_y\right)\left(\tilde{\phi}_{y,0}+\rho_y\tilde{\phi}_{y,1}\right)\\
0 &= \left[(1-\rho_\pi)+\sigma\left(\phi^*_{y,0}+\rho_\pi\phi^*_{y,1}\right)\right]\left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right)+\left[\rho_\pi-\left(\phi^*_{\pi,0}+\rho_\pi\phi^*_{\pi,1}\right)\right]\sigma\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\tilde{\phi}_{\pi,0}+\rho_y\tilde{\phi}_{\pi,1}\right)+\frac{\left(1-\beta\rho_y\right)}{\gamma}\left(\tilde{\phi}_{y,0}+\rho_y\tilde{\phi}_{y,1}\right)\\
0 &= \left(\tilde{\phi}_{\pi,0}+\rho_\pi\tilde{\phi}_{\pi,1}\right)+\frac{\sigma\rho_\pi-\sigma\left(\phi^*_{\pi,0}+\rho_\pi\phi^*_{\pi,1}\right)}{(1-\rho_\pi)+\sigma\left(\phi^*_{y,0}+\rho_\pi\phi^*_{y,1}\right)}\left(\tilde{\phi}_{y,0}+\rho_\pi\tilde{\phi}_{y,1}\right)
\end{aligned}
$$

9. The identified linear combinations

$$
A=\begin{bmatrix}
\gamma & 1-\rho_\pi+\sigma\left(\phi^*_{y,0}+\rho_\pi\phi^*_{y,1}\right)\\
\gamma\rho_y & \rho_\pi\left(1-\rho_\pi+\sigma\left(\phi^*_{y,0}+\rho_\pi\phi^*_{y,1}\right)\right)\\
1-\beta\rho_y & \sigma\left(\rho_\pi-\phi^*_{\pi,0}-\rho_\pi\phi^*_{\pi,1}\right)\\
\left(1-\beta\rho_y\right)\rho_y & \sigma\rho_\pi\left(\rho_\pi-\phi^*_{\pi,0}-\rho_\pi\phi^*_{\pi,1}\right)
\end{bmatrix}
$$

$$
A'\begin{bmatrix}\tilde{\phi}_{\pi,0}\\ \tilde{\phi}_{\pi,1}\\ \tilde{\phi}_{y,0}\\ \tilde{\phi}_{y,1}\end{bmatrix}=0
$$

write this as

$$
\begin{bmatrix}\gamma & \gamma\rho_y & \alpha_2 & \alpha_2\rho_y\\ \alpha_1 & \alpha_1\rho_\pi & \alpha_3 & \alpha_3\rho_\pi\end{bmatrix}\begin{bmatrix}\tilde{\phi}_{\pi,0}\\ \tilde{\phi}_{\pi,1}\\ \tilde{\phi}_{y,0}\\ \tilde{\phi}_{y,1}\end{bmatrix}=0
$$

The nullspace basis is :

$$
\begin{bmatrix}\alpha_2\alpha_3\left(\rho_\pi-\rho_y\right)\\ 0\\ \alpha_1\alpha_2\rho_y-\gamma\alpha_3\rho_\pi\\ \gamma\alpha_3-\alpha_1\alpha_2\end{bmatrix},\begin{bmatrix}\alpha_1\alpha_2\rho_\pi-\gamma\alpha_3\rho_y\\ \gamma\alpha_3-\alpha_1\alpha_2\\ \gamma\alpha_1\left(\rho_y-\rho_\pi\right)\\ 0\end{bmatrix}
$$

Thus, we can identify the two linear combinations

$$
\begin{aligned}
\alpha_2\alpha_3\left(\rho_\pi-\rho_y\right)\tilde{\phi}_{\pi,0}+\left(\alpha_1\alpha_2\rho_y-\gamma\alpha_3\rho_\pi\right)\tilde{\phi}_{y,0}+\left(\gamma\alpha_3-\alpha_1\alpha_2\right)\tilde{\phi}_{y,1} &= 0\\
\left(\alpha_1\alpha_2\rho_\pi-\gamma\alpha_3\rho_y\right)\tilde{\phi}_{\pi,0}+\left(\gamma\alpha_3-\alpha_1\alpha_2\right)\tilde{\phi}_{\pi,1}+\gamma\alpha_1\left(\rho_y-\rho_\pi\right)\tilde{\phi}_{y,0} &= 0
\end{aligned}
$$