

Intermediary Asset Pricing

Zhiguo He Arvind Krishnamurthy *

September 14, 2010

Abstract

We present a model to study the dynamics of risk premia during crises in asset markets where the marginal investor is a financial intermediary. Intermediaries face a constraint on raising equity capital. When the constraint binds, so that intermediaries' equity capital is scarce, risk premia rise to reflect the capital scarcity. We calibrate the model and show that it does well in matching two aspects of crises: the nonlinearity of risk premia during crisis episodes; and, the speed of adjustment in risk premia from a crisis back to pre-crisis levels. We use the model to quantitatively evaluate the effectiveness of a variety of central bank policies, including reducing intermediaries' borrowing costs, infusing equity capital, and directly intervening in distressed asset markets. All of these policies are effective in aiding the recovery from a crisis. Infusing equity capital into intermediaries is particularly effective because it attacks the equity capital constraint that is at the root of the crisis in our model.

JEL Codes: G12, G2, E44

Keywords: Liquidity, LTCM Crisis, Subprime Crisis, Delegated Portfolio Management, Financial Institutions, Asset Pricing, Collateral, Credit.

*Respectively: University of Chicago, Booth School of Business; Northwestern University, Kellogg School of Management and NBER. E-mails: zhiguo-he@chicagobooth.edu, a-krishnamurthy@northwestern.edu. We thank Patrick Bolton, Markus Brunnermeier, Doug Diamond, Andrea Eisfeldt, Vadim Linetsky, Mark Loewenstein, Annette Vissing-Jorgensen, Amir Sufi, Neng Wang, Hongjun Yan and seminar participants at UC-Berkeley, Boston University, UCSB-LAEF conference, University of Chicago, Columbia University, ESSFM Gerzensee, FDIC, University of Maryland, NBER Asset Pricing, NBER Monetary Economics, NY Fed, SF Fed, and Yale for helpful comments.

1 Introduction

The performance of many asset markets – e.g., prices of mortgage-backed securities, corporate bonds, etc. – depend on the financial health of the intermediary sector, broadly defined to include traditional commercial banks as well as investment banks and hedge funds. The subprime crisis and the 1998 hedge fund crisis are two compelling data points in support of this claim.¹ However, traditional approaches to asset pricing ignore intermediation by invoking the assumption that intermediaries’ actions reflect the preferences of their client-investors. With this assumption, the traditional approach treats intermediaries as a “veil,” and instead posits that a representative household is marginal in pricing all assets. Thus, the pricing kernel for the S&P500 stock index is the same as the pricing kernel for mortgage-backed securities. Yet, many crises, such as the subprime crisis and the 1998 episode, play out primarily in the more complex securities that are the province of the intermediary sector. The traditional approach cannot speak to this relationship between financial intermediaries and asset prices. It sheds no light on why “intermediary capital” is important for asset market equilibrium. It also does not allow for a meaningful analysis of the policy actions, such as increasing intermediaries’ equity capital or discount window lending, which are commonly considered during crises.

We offer a framework to address these issues. We develop a model in which the intermediary sector is not a veil, and in which its capital plays an important role in determining asset market equilibrium. We calibrate the model to data on the intermediation sector and show that the model performs well in replicating asset market behavior during crises.

The striking feature of financial crises is the sudden and dramatic increase of risk premia. For example, in the hedge fund crisis of the fall of 1998, many credit spreads and mortgage-backed security spreads doubled from their pre-crisis levels. Our baseline calibration can replicate this dramatic behavior. When intermediary capital is low, losses within the intermediary sector have significant effects on risk premia. However, when capital is high, losses have little to no effect on risk premia. The asymmetry in our model captures the non-linearity that is present in asset market crises. Simulating the model, we find that the average risk premium when intermediaries’ capital constraint is slack is 3.1%. Using this number to reflect a pre-crisis normal level, we find that the probability of the risk premium exceeding 6%, which

¹There is a growing body of empirical evidence documenting the effects of intermediation constraints (such as capital or collateral constraints) on asset prices. These studies include, research on mortgage-backed securities (Gabaix, Krishnamurthy, and Vigneron, 2005), corporate bonds (Collin-Dufresne, Goldstein, and Martin, 2001), default swaps (Berndt, et. al., 2004), catastrophe insurance (Froot and O’Connell, 1999), and index options (Bates, 2003; Garleanu, Pedersen, and Poteshman, 2005).

is twice the “normal” level, is 1.6%. The 1998 episode saw risk premia and Sharpe ratios rise considerably, in the range of $2X$.

Another important feature of financial crises is the pattern of recovery of spreads. In the 1998 crisis, most spreads took about 10 months to halve from their crisis-peak levels to pre-crisis levels. In the subprime crisis, most bond market spreads recovered in about 6 months. As we discuss later in the paper, half-lives of between 6 months and extending over a year have been documented in a variety of asset markets and crisis situations. We note that these types of recovery patterns are an order of magnitude slower than the daily mean reversion patterns documented in the market microstructure literature (e.g., Campbell, Grossman, and Wang, 1993). A common wisdom among many observers is that this recovery reflects the slow movement of capital into the affected markets (Froot and O’Connell, 1999, Berndt, et. al., 2004, Mitchell, Pedersen, and Pulvino, 2007). Our baseline calibration of the model can replicate these speeds of capital movement. We show that simulating the model starting from an extreme crisis state (risk premium of 12%), the half-life of the risk premium back to the unconditional average risk premium is 8 months. From a risk premium of 10%, the half-life is 11 months.

We also use the model as a laboratory to quantitatively evaluate government policies. Beginning from an extreme crisis state with risk premium of 12%, we trace the crisis recovery path conditional on three government policies: (1) Infusing equity capital into the intermediaries during a crisis; (2) Lowering borrowing rates to the intermediary, as with a decrease in the central bank’s discount rate; and, (3) Direct purchase of the risky asset by the government, financed by debt issuance and taxation of households. These three policies are chosen because they are among those undertaken by central banks in practice. Both the equity infusion and risky asset purchase policies have an immediate impact of lowering the risk premium. Moreover, in comparing \$205bn of equity infusion to \$1.8tn of risky asset purchase, we find that the equity infusion is far more effective in reducing the risk premium. This occurs in our model because the friction in the model is an equity capital constraint. Thus infusing equity capital attacks the problem at its heart. The interest rate policy is also effective, uniformly increasing the speed of crisis recovery.

The contribution of our paper is to work out an equilibrium model of intermediation that is dynamic, parsimonious, and can be realistically calibrated. The paper is related to a large literature in banking studying disintermediation and crises (see Diamond and Dybvig (1983), Holmstrom and Tirole (1997), Diamond (1997), and Diamond and Rajan (2005)). We differ from this literature in that our model is dynamic, while much of this literature

is static. The paper is also related to the literature in macroeconomics studying effects of collateral fluctuations on aggregate activity (Kiyotaki and Moore (1997)). In much of the macro literature, equilibrium is derived by log-linearizing around the steady-state. As a result, there is almost no variation in equilibrium risk premia, which does not allow the models to speak to the behavior of risk premia in crises. We solve a fully stochastic model that better explains how risk premia varies as a function of intermediary capital. Brunnermeier and Sannikov (2010) is another recent paper that develops a macroeconomic model that is fully stochastic and links intermediaries' financing position to asset prices. Our paper is also related to the literature on limits to arbitrage studying how impediments to arbitrageurs' trading strategies may affect equilibrium asset prices (Shleifer and Vishny (1997)). One part of this literature explores the effects of margin or debt constraints for asset prices and liquidity in dynamic models (see Gromb and Vayanos (2002), Geanakoplos and Fostel (2008), Adrian and Shin (2010), and Brunnermeier and Pedersen (2008)). Our paper shares many objectives and features of these models. The principal difference is that we study a constraint on raising equity capital, while these papers study a constraint on raising debt financing. Xiong (2001) and Kyle and Xiong (2001) model the effect of arbitrageur capital on asset prices by studying an arbitrageur with log preferences, where risk aversion decreases with wealth. The effects that arise in our model are qualitatively similar to these papers. An advantage of our paper is that intermediaries and their equity capital are explicitly modeled allowing our paper to better articulate the role of intermediaries in crises.² Finally, many of our asset pricing results come from assuming that some markets are segmented and that households can only trade in these markets by accessing intermediaries. Our paper is related to the literature on asset pricing with segmented markets (see Allen and Gale, 1994, Alvarez, Atkeson, and Kehoe, 2002, and Edmond and Weill, 2009).³

Our paper is also related to a companion paper, He and Krishnamurthy (2009). We solve for the optimal intermediation contract in that paper, while we assume the (same) form of contract in the current analysis. That paper also solves for the equilibrium asset prices in closed form, while we rely on numerical solutions in the present paper. On the other hand, that paper has a degenerate steady state distribution which does not allow for a meaningful simulation or the other quantitative exercises we perform in the present paper. In addition, the

²The paper is also related to Vayanos (2005) who studies the effect of an open-ending friction on asset-demand by intermediaries. We study a capital constraint rather than an open-ending friction.

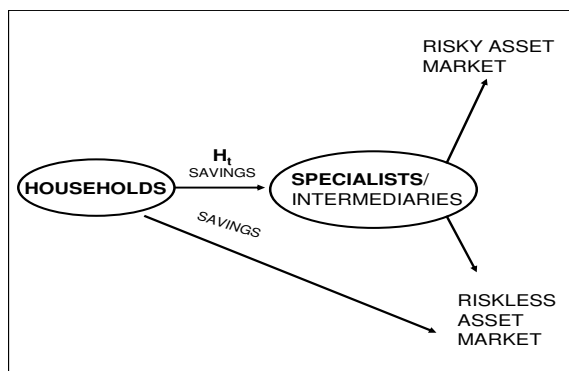
³Our model is also related to the asset pricing literature with heterogeneous agents (see Dumas (1989) and Wang (1996)).

present paper models households with labor income and an intermediation sector which always carries some leverage. Both aspects of the model are important in realistically calibrating the model. However, these same features of the model require us to rely on numerical solutions. Apart from these differences, the analysis in He and Krishnamurthy (2009) provides theoretical underpinnings for some of the assumptions we make in this paper.

The paper is organized as follows. Sections 2 and 3 outline the model and its solution. Section 4 explains how we calibrate the model. Section 5 presents the results of the crisis calibration. Section 6 studies policy actions. Section 7 concludes and is followed by an Appendix with details of the model solution.

2 The Model: Intermediation and Asset Prices

Figure 1: The Economy



This figure depicts the agents in the economy and their investment opportunities.

Figure 1 lays out the building blocks of our model. There is a risky asset that represents complex assets where investment requires some sophistication. In our calibration, we match the risky asset to the market for mortgage-backed securities, as a representative large asset class that fits this description. Investment in the mortgage-backed securities market is dominated by financial institutions rather than households, and sophisticated prepayment modeling is an important part of the investment strategy. The calibration is also appropriate for analyzing the financial crisis that began in 2007, where mortgage-backed securities have a prominent role.

We assume that **households** cannot invest directly in the risky asset market. There is limited market participation, as in Mankiw and Zeldes (1991), Allen and Gale (1994), Basak and Cuoco (1998), or Vissing-Jorgensen (2002). **Specialists** have the knowledge to invest in the risky assets, and unlike in the limited market participation literature, the specialists can invest in the risky asset on behalf of the households. This investment conduit is the intermediary of our model. In our model, the households demand intermediation services while the specialists supply these services. We are centrally interested in describing how this intermediation relationship affects and is affected by the market equilibrium for the “intermediated” risky asset.

We assume that if the household does not invest in the intermediary, it can only invest in a riskless short-term bond. This is clearly counterfactual (i.e. households invest in the S&P 500 index), but simplifies the analysis considerably.

Households thus face a portfolio choice decision of allocating funds between the intermediaries and the riskless bond. The intermediaries accept H_t of the household funds and then allocate their total funds under management between the risky asset and the riskless bond. We elaborate on each of the elements of the model in the next sections.

2.1 Assets

The assets are modeled as in the Lucas (1978) tree economy. The economy is infinite-horizon, continuous-time, and has a single perishable consumption good, which we will use as the numeraire. We normalize the total supply of intermediated risky assets to be one unit. The riskless bond is in zero net supply and can be invested in by both households and specialists.

The risky asset pays a dividend of D_t per unit time, where $\{D_t\}$ follows a geometric Brownian motion,

$$\frac{dD_t}{D_t} = gdt + \sigma dZ_t \quad \text{given } D_0. \tag{1}$$

$g > 0$ and $\sigma > 0$ are constants. Throughout this paper $\{Z_t\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We denote the processes $\{P_t\}$ and $\{r_t\}$ as the risky asset price and interest rate processes, respectively. We also define the total return on the risky asset as,

$$dR_t = \frac{D_t dt + dP_t}{P_t}. \tag{2}$$

2.2 Specialists and intermediation

There is a unit mass of identical specialists who manage the intermediaries in which the households invest. The specialists represent the insiders/decision-makers of a bank, hedge fund, or mutual fund. We collapse all of an intermediary’s insiders into a single agent, following the device of modeling entrepreneur-managers of firms in the corporate finance literature (e.g. Holmstrom and Tirole, 1997). The specialists are infinitely-lived and maximize an objective function,

$$E \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right] \quad \rho > 0; \quad (3)$$

where c_t is the date t consumption rate of the specialist. We consider a CRRA instantaneous utility function with parameter γ for the specialists, $u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$.

Each specialist manages one intermediary. We denote the date t wealth of specialists as w_t and assume that this is wholly invested in the intermediary. We think of w_t as the specialist’s “stake” in the intermediary, possibly capturing financial wealth at risk in the intermediary. Although outside the scope of the model, we may imagine that w_t also captures reputation that is at stake in the intermediary and the future income from being an insider of the intermediary.

We envision the following to describe the interaction between specialists and households. At every t , each specialist is randomly matched with a household to form an intermediary. These interactions occur instantaneously and result in a continuum of (identical) bilateral relationships.⁴ The household allocates some funds H_t to the intermediary. Specialists then execute trades for the intermediary in a Walrasian risky asset and bond market, and the household trades in only the bond market. At $t + dt$ the match is broken, and the intermediation market repeats itself.

Consider one of the intermediary relationships between specialist and household. The specialist manages an intermediary whose total capital is the sum of the specialist’s wealth, w_t , and the wealth that the household allocates to the intermediary, H_t . The specialist makes all investment decisions on this capital and faces no portfolio restrictions in buying or short-selling either the risky asset or the riskless bond. Suppose that the specialist chooses to invest a fraction α_t^I of the portfolio in the risky asset and $1 - \alpha_t^I$ in the riskless asset. Then, the

⁴Why the matching structure instead of a Walrasian intermediation market? In the Walrasian case, when intermediation is supply constrained, specialists charge the households a fee for managing the intermediary that depends on the tightness of the intermediation constraint. In the matching structure the fee is always zero which makes solving the model somewhat easier. Introducing a constant fee into the model is both easy and does not alter results appreciably. See He and Krishnamurthy (2009) where we study the Walrasian case.

return delivered by the intermediary is,

$$\widetilde{dR}_t = r_t dt + \alpha_t^I (dR_t - r_t dt), \quad (4)$$

where dR_t is the total return on the risky asset.

2.3 Intermediary equity capital constraint

The key assumption of our model is that the household is unwilling to invest more than mw_t of funds in the intermediary ($m > 0$ is a constant). That is, if the specialist has one dollar of wealth invested in the intermediary, the household will only invest up to m dollars of his own wealth in the intermediary. He and Krishnamurthy (2009) derive this sort of capital constraint by assuming moral hazard by the specialist. In their model, the household requires that the specialist have a sufficient stake in the intermediary to prevent shirking. Here we adopt the constraint in reduced form.

The wealth requirement implies that the *supply of intermediation* facing a household is at most,

$$H_t \leq mw_t. \quad (5)$$

If either m is small or w_t is small, the household's ability to indirectly participate in the risky asset market will be restricted.

We may interpret the wealth requirement in two ways. First, as noted above, we can think of w_t as the specialist's stake in the intermediary, and this stake must be sufficiently high for households to feel comfortable with their investment in the intermediary. The managers of a hedge fund typically have much of their wealth tied up in terms of the returns of the hedge fund. Hedge fund managers invest some of their wealth in a hedge fund and moreover earn future income based on the returns of the fund. Thus they have a significant stake in the hedge fund's performance which is captured in our model by their ownership share, $\frac{w_t}{w_t + H_t} \geq \frac{1}{1+m}$. The minimum stake requirement ensures that the incentives of the hedge fund's managers and investors are aligned. If a hedge fund loses a lot of money then the capital of the hedge fund will be depleted. In this case, investors will be reluctant to contribute money to the hedge fund, fearing mismanagement or further losses. A hedge fund "capital shock" is one phenomena that we can capture with our model.

The ownership stake interpretation also applies more broadly to the banking sector. Holderness, Kroszner and Sheehan (1999) report that the mean equity ownership of officers and

directors in the Finance, Insurance, and Real Estate sector was 17.4% in 1995. This stake can also be related to the fraction of the intermediary that the specialist owns, $\frac{w_t}{w_t+H_t}$.

Another interpretation, which is more in keeping with regularities in the mutual fund industry, is that the wealth of a specialist summarizes his past success in making investment decisions. Low wealth then reflects poor past performance by a mutual fund, which makes households reluctant to delegate investment decisions to the specialist. The relation between past performance and mutual fund flows is a well-documented empirical regularity (see, e.g., Warther (1995)). As w_t falls, reflecting poor past performance, investors reduce their portfolio allocation to the mutual fund. Shleifer and Vishny (1997) present a model with a similar feature: the supply of funds to an arbitrageur in their model is a function of the previous period's return by the arbitrageur.

Since we adopt constraint (5) in reduced form, we do not take a stand on the interpretation of the constraint. Indeed, in our calibration scenarios, we match the specialist-intermediary to the entire intermediary sector – including hedge funds, banks, and mutual funds. From this standpoint, it is useful that the constraint may be appropriate across a variety of intermediaries.

The novel feature of our model is that w_t , and the supply of intermediation, evolve endogenously as a function of shocks and the past decisions of specialists and households. In both the bank/hedge-fund and the mutual fund example, if the intermediation constraint (5) binds, a fall in w_t causes households to reduce their allocation of funds to intermediaries and invest in the riskless bond. Of course, the risky asset still has to be held in equilibrium. As households indirectly reduce their exposure to the risky asset, via market clearing, the specialist increases his exposure to the risky asset. To induce the specialist to absorb more risk, the risky asset price falls and its expected return rises. This dynamic effect of w_t on the equilibrium is the central driving force of our model. We think it arises naturally when considering the equilibrium effects of intermediation.

We note that both the household and specialist receive the return \widetilde{dR}_t (see (4)) on their contributions to the intermediary; that is, both household and specialist invest in the **equity** of the intermediary. Constraint (5) describes an equity capital constraint on the contribution by the household to the intermediary as a function of the specialist's equity contribution.

Another form of financing constraint that appears important in practice and has been studied by other papers is a **debt or leverage** constraint. In our model, the intermediaries raise equity capital from households as well as borrow by selling (i.e. shorting) riskless bonds. We impose a constraint on raising equity capital but none on borrowing. Denote such borrowing

as B_t . We can imagine a constraint whereby,

$$B_t \leq m^b w_t.$$

Kiyotaki and Moore (1997) in their study of collateral values and business cycles impose a similar constraint. Papers in the asset pricing literature studying margin constraints also impose a similar constraint (see Gromb and Vayanos (2002), Geanakoplos and Fostel (2008), Adrian and Shin (2010), and Brunnermeier and Pedersen (2009)). The margin requirement of these models can be related to $1/m^b$.

We do not study a debt constraint in our model. First, within the logic of the model, any debt that is contracted is always default-free and it thus seems unnatural to impose a debt constraint. That is, B_t is always less than P_t so that if one considers a loan that is collateralized by the asset (i.e. a repo contract), such borrowing carries no default risk. This occurs because the asset price has a continuous sample path so that it is not possible to “jump” into a default state. A model with jumps rather than our Brownian model or a discrete-time model will carry default risk.⁵ Second, equity claims are junior to debt claims and particularly to the collateralized debt claims we observe in practice. Thus any financing constraints are likely to be tighter on equity than debt. The inability of financial institutions to raise equity capital figures prominently in discussions of the subprime crisis.⁶ It is therefore interesting to study a model that drills in particularly on the role of equity capital constraints. Third, as an empirical matter, He, Khang, and Krishnamurthy (2010) document that, during the 2008 crisis period, the commercial banking sector through its access to deposits, discount window financing, as well as other forms of government financing, has essentially faced no constraints on borrowing, while the hedge fund sector has faced such constraints. On the other hand, both the banking sector and the hedge fund sector have had limited equity capital. This suggests that equity capital constraints are more widespread than debt constraints. It would

⁵It is likely that the constrained borrowing we observe in practice is not on 100% safe debt but on risky debt. The fact that repo haircuts vary across the riskiness of the underlying collateral suggests that the repo debt is only partly collateralized. Given standard spanning arguments, we can think of risky debt as a combination of safe debt and a position in the underlying risky asset. That is define risky debt, \tilde{B}_t as being composed of safe debt B_t and a position δ in the risky asset. Investors may ration their supply of risky debt because they are rationing their exposure δ to the risky asset. The equity capital constraint we have imposed is directly a restriction on investors willingness to own exposure to the risky asset; i.e. it is qualitatively similar to a restriction on δ . In this sense, our model captures aspects of constrained risky-debt financing.

⁶In He and Krishnamurthy (2009), we allow for all forms of contracts and derive an optimal contract that places a constraint on equity capital contributions but no constraint on debt contributions.

be interesting in future work to study a model with both debt and equity capital constraints where different parts of the intermediary sector are modeled to have different constraints. Indeed, it is likely, as we discuss later in the paper, that the model's fit can be improved with such an embellishment.

2.4 Specialist/intermediary decision

The specialist chooses his consumption rate and the portfolio decision of the intermediary to solve,

$$\max_{\{c_t, \alpha_t^I\}} E \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right] \quad s.t. \quad dw_t = -c_t dt + w_t r_t dt + w_t \left(\widetilde{dR}_t(\alpha_t^I) - r_t dt \right). \quad (6)$$

We can also rewrite the budget constraint in terms of the underlying return:

$$dw_t = -c_t dt + w_t r_t dt + \alpha_t^I w_t (dR_t - r_t dt).$$

Note that α_t^I is effectively the specialist's portfolio share in the risky asset.

2.5 Households: The demand for intermediation

We model the household sector as an overlapping generation (OG) of agents. This keeps the decision problem of the household fairly simple.⁷ On the other hand, we enrich the model to include household labor income and introduce heterogeneity within the household sector. Without household income it is possible to reach states where the household sector vanishes from the economy, rendering our analysis uninteresting (see, for example, Dumas (1989) and Wang (1996) for more on this problem in two-agent models). We also introduce labor income to more realistically match the consumption-savings profile of households. Likewise, heterogeneity within the household sector is useful in realistically calibrating the model.

For the sake of clarity in explaining the OG environment in a continuous time model, we index time as $t, t + \delta t, t + 2\delta t, \dots$ and consider the continuous time limit when δt is of order dt . A unit mass of generation t agents are born with wealth w_t^h and live in periods t and $t + \delta t$. They maximize utility:

$$\rho \delta t \ln c_t^h + (1 - \rho \delta t) E_t[\ln w_{t+\delta t}^h]. \quad (7)$$

⁷Note the specialists are infinitely lived while households are modeled using the OG structure. As we will see, specialists play the key role in determining asset prices. Our modeling ensures that their choices reflect the forward-looking dynamics of the economy. We treat households in a simpler manner for tractability reasons. We deem the cost of the simplification to be low since households play a secondary role in the model.

c_t^h is the household's consumption in period t and $w_{t+\delta t}^h$ is a bequest for generation $t + \delta t$. Note that both utility and bequest functions are logarithmic.

In addition to wealth of w_t^h , we assume that generation t households receive labor income at date t of $l D_t \delta$. $l > 0$ is a constant and D_t is the dividend on the risky asset at time t . Labor income is assumed proportional to dividends in order to preserve some useful homogeneity properties of the equilibrium.

It is easy to verify that as $\delta t \rightarrow dt$ in the continuous time limit, the household's consumption rule is,

$$c_t^h = \rho w_t^h. \tag{8}$$

In particular, note that the labor income does not affect the consumption rule because the labor income flow is of order dt . Interpreting $\rho > 0$ as the household's rate of time preference, we note that this is the standard consumption rule for logarithmic agents. The household is "myopic" and his rule does not depend on his investment opportunity set.

A household invests its wealth from t to $t+\delta$ in financial assets. As noted earlier, households are not directly able to save in the risky asset and can only directly access the riskless bond market. We assume that the household can choose any positive level of bond holdings when saving in the riskless bond (note that short-selling of the bond is ruled out). The household must use an intermediary when accessing the risky asset market.

We consider a further degree of heterogeneity in the intermediation investment restriction. We assume that a fraction λ of the households can ever only invest in the riskless bond. The remaining fraction, $1 - \lambda$, may enter the intermediation market and save a fraction of their wealth with intermediaries which indirectly invest in the risky asset on their behalf. We refer to the former as "debt households" and the latter as "risky asset households."⁸

The heterogeneity among households is realistic. Clearly, there are many households that only save in a bank account. In the literature cited earlier on limited market participation, all households are "debt households." The demand for intermediation in our model stems from the risky asset households. Introducing this degree of heterogeneity allows for a better model calibration.

⁸The wealth of the debt household and risky asset household evolve differently between t and $t + \delta$. We assume that this wealth is pooled together and distributed equally to all agents of generation $t + \delta$. The latter assumption ensures that we do not need to keep track of the distribution of wealth over the households when solving for the equilibrium of the economy.

2.6 Household decisions

To summarize, a debt and risky asset household are born at generation t with wealth of w_t^h . The households receive labor income and choose a consumption rate of ρw_t^h . They also make savings decisions, respecting the restriction on their investment options. The debt household's consumption decision, given wealth of w_t^h , is described by (8). The savings decision is to invest w_t^h in the bond market at the interest rate r_t . The risky asset household's consumption is also described by (8). His portfolio decision is how much wealth to allocate to intermediaries. We denote $\alpha_t^h \in [0, 1]$ as the fraction of the household's wealth in the intermediary and recall that the intermediary's return is \widetilde{dR}_t in (4). The remaining $1 - \alpha_t^h$ of household wealth is invested in the riskless bond and earns the interest rate of $r_t dt$. The risky asset household chooses α_t^h to maximize (7). Given the log objective function, this decision solves,

$$\max_{\alpha_t^h \in [0,1]} \alpha_t^h E_t[\widetilde{dR}_t] - \frac{1}{2} (\alpha_t^h)^2 \text{Var}_t[\widetilde{dR}_t] \quad s.t. \quad \alpha_t^h (1 - \lambda) w_t^h \equiv H_t \leq m w_t. \quad (9)$$

Note the constraint here, which corresponds to the intermediation constraint we have discussed earlier.

Given the decisions by the debt household and the risky asset household, the evolution of w_t^h across generations is described by,

$$dw_t^h = (lD_t - \rho w_t^h)dt + w_t^h r_t dt + \alpha_t^h (1 - \lambda) w_t^h (\widetilde{dR}_t - r_t dt). \quad (10)$$

2.7 Equilibrium

Definition 1 *An equilibrium is a set of price processes $\{P_t\}$ and $\{r_t\}$, and decisions $\{c_t, c_t^h, \alpha_t^I, \alpha_t^h\}$ such that,*

1. *Given the price processes, decisions solve the consumption-savings problems of the debt household, the risky asset household (9) and the specialist (6);*
2. *Decisions satisfy the intermediation constraint of (5);*
3. *The risky asset market clears:*

$$\frac{\alpha_t^I (w_t + \alpha_t^h (1 - \lambda) w_t^h)}{P_t} = 1; \quad (11)$$

4. *The goods market clears:*

$$c_t + c_t^h = D_t (1 + l). \quad (12)$$

Given market clearing in risky asset and goods markets, the bond market clears by Walras' law. The market clearing condition for the risky asset market reflects that the intermediary is the only direct holder of risky assets and has total funds under management of $w_t + \alpha_t^h(1 - \lambda)w_t^h$, and the total holding of risky asset by the intermediary must equal the supply of risky assets.

Finally, an equilibrium relation that proves useful when deriving the solution is that,

$$w_t + w_t^h = P_t.$$

That is, since bonds are in zero net supply, the wealth of specialists and households must sum to the value of the risky asset.

3 Solution

We outline the main steps in deriving the solution in this section. For detailed derivations, see the Appendix A. We begin with an example that illustrates the main features of our model and helps in understanding the steps in the solution.

3.1 Example

Suppose that $m = 1$ and $\lambda = 0$. Moreover, suppose we are in a state where $w_t = 100$ and $w_t^h = 200$. Then it is clear that since $mw_t < w_t^h$, this is a state where intermediation is constrained by (5). Since the riskless asset is in zero net supply, the value of the risky asset is equal to the sum of w_t and w_t^h (i.e. 300). Suppose that households saturate the intermediation constraint by investing 100 in intermediaries. Then intermediaries have total equity contributions of 200 (the households' 100 plus the specialists' w_t). Since intermediaries hold all of the risky asset worth 300, their portfolio share in the risky asset must be equal to 150%. Their portfolio share in the bond is -50% . That is, the intermediary holds a levered position in the risky asset. The household's portfolio shares are $0.5 \times 150\% = 75\%$ in risky asset; and, 25% in debt. The households and specialists have different portfolio exposures to the risky asset. But since the specialist drives the pricing of the risky asset, risk premia must adjust to make the 150% portfolio share optimal.

From this situation, suppose that dividends on the risky asset fall. Then, since the specialists are more exposed to the risky asset than households, w_t falls relative to w_t^h . The shock then further tightens the intermediation constraint, which creates an amplified response to the shock.

Contrast this situation with one in which there is no intermediation constraint. Suppose that households invest all of their wealth with the intermediaries. Since intermediaries now have 300 and the risky asset is worth 300, the portfolio share of both specialists and households is equal to 100%. Both agents share equally in the asset's risk and shocks do not affect the distribution of wealth between the agents.

3.2 State Variables and Specialists' Euler Equation

We look for a stationary Markov equilibrium where the state variables are (y_t, D_t) , where $y_t \equiv \frac{w_t^h}{D_t}$ is the dividend scaled wealth of the household. As the example illustrates, the intermediation frictions depend on the distribution of wealth between households and specialists. We capture this relative distribution by y_t .

As standard in any CRRA/GBM economy, our economy is homogeneous in dividends D_t . We conjecture that the equilibrium risky asset price is,

$$P_t = D_t F(y_t), \tag{13}$$

where $F(y)$ is the price/dividend ratio of the risky asset.

Now we use the agents' optimal decisions and market clearing conditions to derive the equation for F . While the household faces investment restrictions on his portfolio choices, the specialist (intermediary) is unconstrained in his portfolio choices. This important observation implies that the specialist is always the marginal investor in determining asset prices, while the household may not be. Standard arguments then tell us that we can express the pricing kernel in terms of the specialist's equilibrium consumption process.

We have noted in (8) that the household's optimal consumption given w_t^h is $c_t^h = \rho w_t^h$, which we can rewrite as $c_t^h = \rho y_t D_t$. Now the market clearing condition for goods (from (12)) is,

$$c_t + \rho y_t D_t = D_t(1 + l).$$

Thus, in equilibrium, the specialist consumes:

$$c_t = D_t(1 + l - \rho y_t). \tag{14}$$

We thereby express specialist consumption as a function of the state variables D_t and y_t .

Optimality for the specialist gives us the standard consumption-based asset pricing rela-

tions (Euler equation):⁹

$$-\rho dt - \gamma E_t \left[\frac{dc_t}{c_t} \right] + \frac{1}{2} \gamma (\gamma + 1) Var_t \left[\frac{dc_t}{c_t} \right] + E_t [dR_t] = \gamma Cov_t \left[\frac{dc_t}{c_t}, dR_t \right]; \quad (15)$$

and for interest rate, we have

$$r_t dt = \rho dt + \gamma E_t \left[\frac{dc_t}{c_t} \right] - \frac{\gamma(\gamma + 1)}{2} Var_t \left[\frac{dc_t}{c_t} \right]. \quad (16)$$

Using (14) and (13), we can express dR_t and $\frac{dc_t}{c_t}$ as a function of the derivatives of $F(y)$, and the unknown drift and diffusion of y_t . These unknown drift and diffusion will depend on the households' equilibrium portfolio choices, which is the focus of the next section. Combining these results, we arrive at a differential equation that must be satisfied by $F(y)$ (see Appendix A).

3.3 Dynamics of Household Wealth

Given the wealth dynamics of the household in (10) and the intermediary return $\widetilde{dR}_t - r_t dt = \alpha_t^I (dR_t - r_t dt)$, we have

$$dw_t^h = (lD_t - \rho w_t^h) dt + w_t^h r_t dt + (\alpha_t^h \alpha_t^I) (1 - \lambda) w_t^h (dR_t - r_t dt).$$

We now determine the household's exposure to the risky asset return $(\alpha_t^h \alpha_t^I) (1 - \lambda)$. First, note that when the intermediation constraint of equation (5) binds, the household choice must satisfy

$$\alpha_t^{h,const} (1 - \lambda) w_t^h = m w_t, \quad \text{which implies,} \quad \alpha_t^{h,const} = \frac{m (F(y) - y)}{(1 - \lambda) y}. \quad (17)$$

That is, the binding constraint pins down the household's portfolio share in the intermediary. Moreover, since all risky assets are held through the intermediary, the equilibrium market clearing condition (11) gives,

$$\frac{\alpha_t^{I,const} (w_t + m w_t)}{P_t} = 1.$$

Using the fact that $w_t + w_t^h = P_t$, we find,

$$\alpha_t^{I,const} = \frac{1}{1 + m} \frac{F(y)}{F(y) - y}. \quad (18)$$

The logic in arriving at this expression is the same as in the example.

When the intermediation constraint does not bind, the household is unconstrained in choosing α_t^h . We make an assumption that implies that $\alpha_t^h = 1$ in this case:

⁹The Euler equation is a necessary condition for optimality. In Appendix B, we prove sufficiency.

Parameter Assumption 1 *We focus on parameters of the model such that in the absence of any portfolio restrictions, the risky asset household will choose to have at least 100% of his wealth invested in the intermediary, i.e., $\alpha_t^h = 1$.*

Although we are unable to provide a precise mathematical condition for this parameter restriction, in our calibration it appears that $\gamma \geq 1$ is a sufficient condition. Loosely speaking, if the specialist is more risk averse than the household, the household will hold more risky assets than the specialist. But given market clearing in the risky asset market, the specialist always holds more than 100% of his wealth in the risky asset. Recall that we assume that the household cannot short bonds. Thus, the household allocates the maximum of 100% of his wealth to the intermediary. Using the market clearing condition for risky assets, we find,

$$\alpha_t^{I,unconst} = \frac{F(y)}{F(y) - \lambda y}. \quad (19)$$

3.4 Constraint Threshold

We now characterize the conditions under which the intermediation constraint binds. Setting $\alpha_t^h = 1$ in (5) yields that the constraint binds when,

$$(1 - \lambda)w_t^h \geq mw_t.$$

Using $w_t^h + w_t = P_t$, we rewrite the inequality to find an expression that gives a cutoff for the constrained states:

$$y^c = \frac{m}{1 + m - \lambda} F(y^c).$$

This equation has a unique solution in all of our parameterizations.

In summary, when $y < y^c$, the intermediation constraint is binding, and we have the expressions for α_t^h and α_t^I as in (17) and (18). When $y > y^c$, the household chooses $\alpha_t^h = 1$ and α_t^I is given by (19).

3.5 Boundary Condition

The model has a natural upper boundary condition on y that is determined by the goods market clearing condition. Since

$$c_t = D_t(1 + l - \rho y_t),$$

and the specialist's consumption c_t must be positive, y_t has to be bounded by

$$y^b \equiv \frac{1+l}{\rho}.$$

In Appendix B, we show that y^b is an entrance-no-exit boundary, and that y_t never reaches y^b .

On an equilibrium path in which y approaches y^b , the specialist's equilibrium consumption c goes to zero. Since the specialist's wealth is $w = D(F(y) - y)$, one natural guess for the boundary condition at this singular point y^b is

$$F(y^b) = y^b. \tag{20}$$

In words, when the specialist's consumption approaches zero, his wealth also converges to zero. In the argument for verification of optimality of the specialist's equilibrium strategy which is detailed in Appendix B, we see that this condition translates to the transversality condition for the specialist's budget equation. Therefore the boundary condition (20) is sufficient for the equilibrium presented in this paper to be well-defined.

4 Calibration

Table 1 provides data on the main intermediaries in the US economy. Households hold wealth through a variety of intermediaries including banks, retirement funds, mutual funds, and hedge funds.¹⁰

4.1 Choice of m

The main challenge in the calibration is that the model treats the entire intermediary sector as a group of identical institutions, while it is clear from Table 1 that there is functional heterogeneity across the modes of intermediation. In particular, some of the intermediaries, such as mutual and pension funds, are financed only by equity, while some intermediaries, such as banks or hedge funds always carry leverage. Note that in our model the capital structure of the intermediary plays a central role in asset price determination. When the intermediation constraint (5) binds, losses among intermediaries lead households to reduce their equity exposure to these intermediaries. If the intermediaries scale down their asset

¹⁰We need to be careful in interpreting these numbers because there is some amount of double counting – i.e. pension funds invest in hedge funds.

Table 1: Intermediation Data (\$ Billions) ^a

Group	Assets	Debt	Debt/Assets
Commercial banks	11,800	10,401	0.88
S&L and Credit Unions	2,574	2,337	0.91
Property & Casualty Insurance	1,381	832	0.60
Life Insurance	4,950	4,662	0.94
Private Pensions	6,391	0	0.00
State & Local Ret Funds	3,216	0	0.00
Federal Ret Funds	1,197	0	0.00
Mutual Funds (excluding Money Funds)	7,829	0	0.00
Broker/Dealers	2,519	2,418	0.96
Hedge Funds	6,913	4,937	0.71

^a Most data are from the Flow of Funds March 2010 Level Tables, corresponding to the year 2007. The broker/dealer and hedge fund total assets are as computed in He, Khang, and Krishnamurthy (2010), who use data from SEC filings for the broker/dealer sector and data from Barclay's Hedge for the hedge fund sector. We assume that the average broker/dealer runs a leverage of 25, based on Adrian and Shin (2010). We assume the average hedge fund leverages up its capital base 3.5 times (taken from McGuire, Remolona and Tsatsaronis (2005))

holdings proportionately, the asset market will not clear – i.e. the intermediary sector's assets still have to be held in equilibrium. In the model, the equilibrium is one where the [identical] intermediaries take on debt and hold a riskier position in the asset. Asset prices are then set by the increased risk/leverage considerations of the intermediaries. In practice, if households withdraw money from mutual funds, then mutual funds do not take on debt. Rather, they reduce their holdings of financial assets and some other entity buys their financial assets. In practice, the other entity may be a trading desk at a bank or a hedge fund that temporarily provides liquidity to the mutual fund. It may be that the buyers have excess equity capital in which case the purchase can occur without specialists having to increase their risk exposure/leverage and therefore without equilibrium asset prices adjusting appreciably. This situation corresponds to the unconstrained region of the model. However, if equity capital is constrained, as in the model's constrained region, then the purchase will be financed by raising debt, increasing leverage, and increasing risk concentration. In this case, asset prices will be affected and driven by the limited equity capital of the buyers (i.e. banks/hedge funds).

The m of the model parameterizes the equity capital constraint of the intermediaries. From the discussion of the preceding paragraph, we see that to model asset price behavior we want m to correspond to the equity capital constraints of banks/hedge funds rather than features of the broad intermediary sector. This is because it is the marginal pricing condition of these

intermediaries that is most relevant during a liquidation crisis.

We set m equal to 4, which matches both ownership data of banks and compensation data from hedge funds. Holderness, Kroszner and Sheehan (1999) report that the mean equity ownership of officers and directors in the Finance, Insurance, and Real Estate sector was 17.4% in 1995. This translates to an m of $4.7(= \frac{1-0.174}{0.174})$. We also present an $m = 8$ case to provide a sense as to the sensitivity of the results to the choice of m . Hedge fund contracts typically pay the manager 20% of the fund's return in excess of a benchmark, plus 1 – 2% of funds under management (Fung and Hsieh, 2006). The choice of m dictates how much of the return of the intermediary goes to the specialist ($\frac{1}{1+m}$) and how much go to equity investors ($\frac{m}{1+m}$). A value of $m = 4$ implies that the specialist's share $1/5 = 20\%$. The 20% that is common in hedge fund contracts is an option contract so it is not a full equity stake as in our model, suggesting that perhaps we should use a larger value of m . However, to balance this, note that the 1 – 2% fee is on funds under management and therefore grows as the fund is successful and garners more inflows. We thus settle on a value of $m = 4$ as representative, in a linear scheme, of the payoff structure of the hedge fund.

4.2 Choice of λ

As noted above, m only plays a role in the constrained region of the model. In practice, we can see from Table 1 that the intermediary sector always has some leverage, whether in a crisis or not. It is important to match leverage in the unconstrained region because leverage affects how dividend shocks get magnified and hence how the state transits from unconstrained to constrained region.

We choose $\lambda = 0.6$ to match leverage in the unconstrained region. In this region, we interpret the model's single intermediary as being an amalgam of all the intermediaries in Table 1. Within the model, when $\lambda > 0$ some households only demand debt, and the intermediaries supply the debt and thereby achieve leverage even when intermediation is not constrained. Across all of the intermediaries of Table 1, the Total Debt/Total Assets ratio is 0.52. Setting $\lambda = 0.6$ in the model produces an average debt-to-asset ratio in the unconstrained region of 0.50, and an unconditional average debt-to-asset ratio of 0.55.

4.3 σ and g

We calibrate the intermediated asset to the market for mortgage-backed securities (MBS) as a representative large intermediated asset class. The Securities Industry and Financial

Markets Association (SIFMA) reports that the total outstanding MBS securities (Agency-backed MBS, private-label MBS, commercial MBS) totaled \$8.9tn in 2007. SIFMA reports that the outstanding amount of asset-backed securities (auto, credit card, etc.) totaled \$2.5tn in 2007. We are unaware of data that allow us to know precisely who holds these securities. However, the pattern of losses as reported by financial institutions in the subprime crisis, and most analyses of losses (e.g., see the IMF's Global Financial Stability Report of October 2008) suggests these securities are mostly held in intermediary portfolios.

The Barclays Capital U.S. MBS Index (formerly the Lehman Brothers U.S. MBS index) tracks the return on the universe of Agency-backed MBS from 1976 onwards. The annual standard deviation of the excess return of this index over the Treasury bill rate, using data from 1976 to 2008, is 8.1%. Note that this index measures the returns on Agency-backed MBS which is the least risky (although largest) segment of the MBS market. As another benchmark, the annual standard deviation of the excess return on Barclays index of commercial MBS over the period 1999 (i.e. inception of the index) to 2008 is 9.6%.

We choose σ to be 9%. With this choice, the standard deviation of the excess return on the intermediated asset in our model is 9.2%. This number is in the range between the low risk Agency MBS and the higher risk commercial MBS.¹¹

We choose $g = 1.84\%$. We would expect that the payouts on mortgage assets should grow

¹¹Our choice of $\sigma = 9\%$ is an order of magnitude higher than aggregate consumption volatility of close to 3%. In standard general equilibrium approaches to asset pricing, exemplified by Campbell and Cochrane (1999) or Barberis, Huang, and Santos (2001), models assume a representative agent whose consumption is equal to NIPA aggregate consumption and price a payoff with a dividend stream that matches properties of aggregate stock market dividends.

The marginal investor in our model is the specialist-intermediary rather than a representative agent because intermediaries are not a veil. As our analysis shows, the specialist's marginal utility is endogenously affected by fluctuations in the value of assets that the specialist holds. Thus, we do not exogenously specify the marginal investor's consumption process based on aggregate consumption, but endogenously derive the joint behavior of specialist consumption and the prices of intermediated assets. For this reason, we choose the volatility of the risky asset's dividends to match those of financial payoffs rather than that of aggregate consumption. Indeed, we see the endogenous relationship between financial wealth fluctuations and the pricing kernel as an important reason to model intermediaries rather than treat them as a veil.

Finally, in principle it seems possible to reconcile the low aggregate consumption volatility we observe in practice with the 9% dividend volatility of the model by assuming that the household sector has income from other assets (i.e stock market dividends) and labor income, and that this income is weakly correlated with the returns from mortgage-backed securities. Unfortunately, such a model will no longer be homogeneous with respect to D_t which will considerably complicate the analysis.

with the economy. We set g based on the growth of dividends in the stock market, taking such growth to reflect the general rate of cash-flow growth in the economy. The choice of g has a minor effect on results. On the other hand, σ is critical because it is closely related to the amount of risk borne by the specialist and the volatility of the intermediary pricing kernel.

Table 2: Parameters

Panel A: Intermediation		
m	Intermediation multiplier	4
λ	Debt ratio	0.6
Panel B: Preferences and Cashflows		
g	Dividend growth	1.84%
σ	Dividend volatility	9%
ρ	Time discount rate	5%
γ	RRA of specialist	2
l	Household labor income ratio	1.3

4.4 γ , l , and ρ

We choose $\gamma = 2$ as risk aversion of the specialist. This choice of γ produces an average excess return on the intermediated asset of 3.4%. Over the 1976 to 2008 period, the average excess return on the Barclay's Agency MBS index was 2.6%. Over the 1999 to 2008 period, the return on the commercial MBS index was 0.32%. However, the latter sample is quite short and heavily weighted by a large -22.9% return in 2008. Note also that allowing for $\gamma > 1$ for the specialist allows us to capture dynamic hedging effects that would be absent if set the specialist to have log preferences to match with the household.

We choose l to match the income profile of a typical household. In our model, households receive expected capital and dividend income of $E \left[w_t r_t dt + (1 - \lambda) \alpha_t^h \left(\widetilde{dR}_t - r_t dt \right) \right]$ and expected labor income of $E[lD_t dt]$. We set $l = 1.3$ which produces a capital income to total income share of 33.7%. Parker and Vissing-Jorgensen (2010) report that the average capital income to total income for households over the period from 1982 to 2006 was 32%.

We choose $\rho = 0.05$. This choice produces an average riskless interest rate of 0.62%, which is in the range of typical numbers in the literature. Finally, our parameter choices are also dictated by the restriction that, $\rho + g(\gamma - 1) - \frac{\gamma(\gamma-1)\sigma^2}{2} - \frac{l\gamma\rho}{1+l} > 0$. This restriction is necessary to ensure that the economy is well-behaved at $t = \infty$ (see Appendix A).

4.5 Numerical method

We present numerical solutions based on the calibration of Table 2. We use one of *MATLAB*'s built-in ODE solvers to derive solutions for $F(y)$, μ_y , and σ_y . Further details are provided in the Appendix A.

With these solutions in hand, we numerically simulate the model to obtain the steady state distribution of the state variable y as well as a number of asset price measurements that we report in the next sections. We begin the economy at a state ($y_0 = y^c$, $D_0 = 1$) and simulate the economy for 5000 years. That is we obtain a sequence of independent draws from the normal distribution and use these draws to represent innovations in our shock process Z_t . The path of Z_t can then be mapped into a path of the state variable. We compute the time-series averages of a number of relevant asset price measurements from years 1000 to 5000 of this sample. The simulation unit is monthly, and based on those monthly observations we compute annual averages. We repeat this exercise 5000 times, averaging across all of the simulated Z_t paths. We find that changing the starting value y_0 does not affect the computed distribution or any of the asset price measurements, indicating that the distribution truly represents the steady state distribution of the economy.

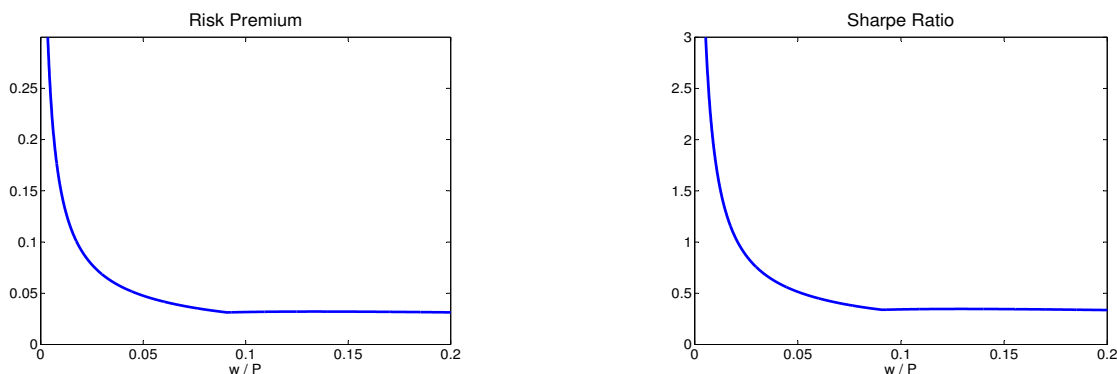
5 Crisis Behavior

5.1 Risk Premium and Sharpe Ratio

Figure 2 graphs the risk premium and Sharpe ratio for the calibration of Table 2 as a function of the specialist wealth relative to the value of the risky asset (w/P). The latter ratio can be interpreted as the inside capital of the intermediation sector as a percentage of the assets held by the intermediation sector. Even though we solve our model based on the household's scaled wealth $y = w^h/D$, we decide to illustrate our results using w/P in order to more clearly discuss the role of intermediation capital.

The prominent feature of our model, clearly illustrated by the graphs, is the asymmetric behavior of the risk premium and Sharpe ratio. The right hand side of the graphs represent the unconstrained states of the economy, while the left hand side represent the constrained states. The cutoff for the constrained region in the figures is 0.091. In words, the constrained region arises when specialists own [less than] equity equivalent to 9.1% of the assets held by intermediaries. Note that this number refers to the equity ownership of the entire interme-

Figure 2: Risk Premium and Sharpe Ratio



Risk premium (left panel) and Sharpe ratio (right panel) are graphed against w/P , the specialist wealth as a percentage of the assets held by the intermediation sector. Parameters are those given in Table 2.

diation sector; there may be some sectors where the specialists own far less than 9.1%, and some where the specialists own more. Risk premia and Sharpe ratio rise as specialist wealth falls in the constrained region, while being relatively constant in the unconstrained region.

This asymmetric behavior is intuitively what one would expect from the model: the model's intermediation constraint is by its nature asymmetric, and binding only when specialist wealth is low. To sharpen understanding of the mapping between the constraint and risk premia, consider the following calculation. As noted above, the pricing kernel in our model can be expressed in terms of the specialist's consumption. Thus, the risk premium on the risky asset is equal to:

$$\gamma \operatorname{cov}_t \left(\frac{dc_t}{c_t}, dR_t \right)$$

To a first-order approximation, the volatility of the specialist's consumption growth is equal to the volatility of the return on his wealth (the approximation is exact if $\gamma = 1$). Thus,

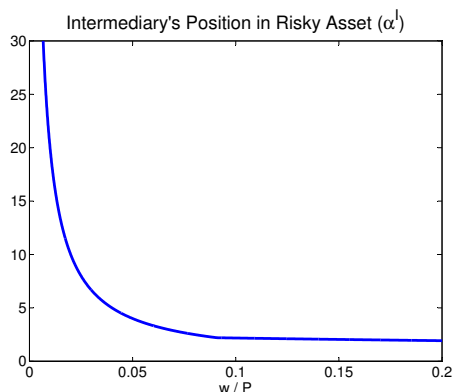
$$\operatorname{var}_t \left(\frac{dc_t}{c_t} \right) \approx (\alpha_t^I)^2 \operatorname{var}_t(dR_t),$$

where α_t^I is the portfolio exposure to the risky asset in the intermediary's (and specialist's) portfolios. Therefore, the risk premium is approximately,

$$\gamma (\alpha_t^I)^2 \operatorname{var}_t(dR_t).$$

In our model, the variance of returns is roughly constant as a function of state (see the discussion of this point below). Most of the action in the risk premium comes from the changing α_t^I . We have noted before that in the constrained region, as households withdraw from intermediaries and limit their participation in the risky asset market, the specialists increase their exposure to the risky asset (see equation (18)). This dynamic, driven through α_t^I , explains the behavior of the risk premium. Figure 3 graphs α^I . We note the close correspondence between this graph and those in Figure 2.

Figure 3: Portfolio Holdings



The intermediary's portfolio share in the risky asset (α^I) is graphed against w/P .

An interesting point of comparison for our results is to the literature on state-dependent risk premia, notably, Campbell and Cochrane (1999), Barberis, Huang, and Santos (2001), and Kyle and Xiong (2001). In these models, as in ours, the risk premium is increasing in the adversity of the state. Campbell and Cochrane and Barberis, Huang, and Santos modify the utility function of a representative investor to exhibit state-dependent risk aversion. We work with a standard CRRA utility function, but generate state dependence endogenously as a function of the frictions in the economy. For empirical work, our approach suggests that measures of intermediary capital/capacity will help to explain risk premia. In this regard, our model is closer in spirit to Kyle and Xiong who generate a risk premium that is a function of “arbitrageur” wealth. The main theoretical difference between Kyle and Xiong and our

model is that the wealth effect in their model comes from assuming that the arbitrageur has log utility, while in our model it comes because the intermediation constraint is a function of intermediary capital. For empirical work, our model suggests that measures of intermediary capital will explain risk premia. One notable distinction of our model is the sharp asymmetry of our model's risk premia: a muted dependence on capital in the unconstrained region and a strong dependence in the constrained region. In Kyle and Xiong, the log utility assumption delivers a risk premium that is a much smoother function of arbitrageur wealth. Plausibly, to explain a crisis episode, one needs the type of asymmetry delivered by our model.

5.2 Discussion: Leverage and Heterogeneity

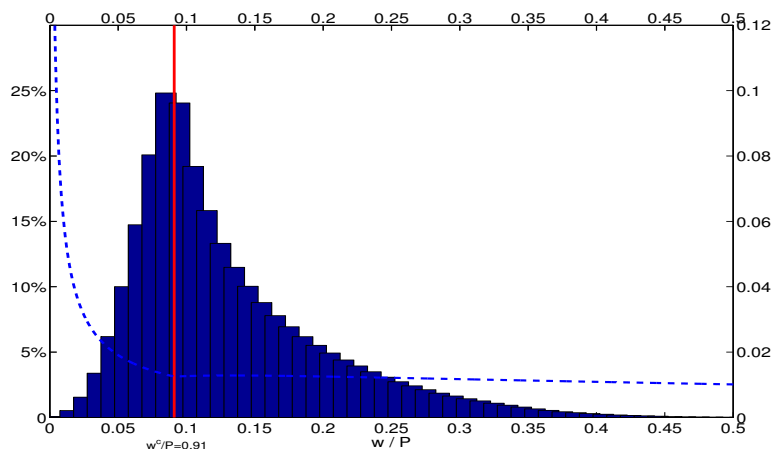
Figure 3 also shows that the rise in the risk premium in the constrained region is closely related to the rise in leverage of the intermediary sector. In practice, many intermediary sectors during a crisis reduce leverage, while other sectors increase leverage. As with our earlier discussion of calibration, there is heterogeneity within the intermediation sector that our single intermediary model cannot capture. Adrian and Shin (2010) document that the leverage of the broker/dealer sector is procyclical, suggesting that it falls during recessions and crises. He, Khang, and Krishnamurthy (2010) document that in the period from the fourth quarter of 2007 to the first quarter of 2009, spanning the worst episode of the subprime crisis, the hedge fund and broker/dealer sector shed assets, consistent with the deleveraging evidence of Adrian and Shin as well as theoretical papers modeling leverage constraints (see Gromb and Vayanos, 2002, Geanakoplos and Fostel, 2008, and Brunnermeier and Pedersen, 2008). He, Khang, and Krishnamurthy show that the commercial banking sector increased asset holdings over this period significantly. Moreover, the leverage of the top 19 commercial banks sector rises from 10.4 at the end of 2007 to near 30 at the start of 2009. He, Khang, and Krishnamurthy suggest that the differential behavior of the commercial banking sector vis-a-vis the hedge fund sector is that the former had access to government-backed debt financing, which aided their leverage growth. The differential behavior of the banking sector in 2008 is reflective of a broader pattern of reintermediation during financial downturns, as documented by Gatev and Strahan (2006) and Pennacchi (2006). Importantly for the present analysis, in accord with our model the intermediaries that are the buyers during the crisis (i.e. banks) do so by borrowing and increasing leverage. Our model does not capture the other aspect of this process, as reflected in the behavior of the hedge fund and broker/dealer sector, that other parts of the financial sector reduce asset holdings and leverage. It would be interesting to

build a model with heterogeneity within the intermediary sector to more fully address these patterns.

5.3 Steady State Risk Premia

Quantitatively, as one can read from Figure 2, the calibration produces an average risk premium in the unconstrained region of approximately 3%. The numbers for the risk premium are higher in the constrained region; however, without knowing the probability that a given specialist-wealth state may occur, it is not possible to interpret a statement about how much higher. To provide some sense for the values of the risk premium we may be likely to observe in practice, we simulate the model as described in Section 4.5 and compute the equilibrium probability of each state. The resulting steady state distribution over the specialist wealth as a percentage of the assets held by the intermediation sector (w/P) is graphed in Figure 4. Also superimposed on the figure in a dashed line is the risk premium from the previous graph.

Figure 4: Steady State Distribution



The steady state distribution of w/P is graphed. The vertical line gives the state where the intermediation constraint starts binding ($w/P = 0.091$). The dashed line graphs the risk premium in order to illustrate the actual range of variation of the risk premium. Risk premium is indicated on the left scale, while the distribution is indicated on the right scale.

There are two forces driving the center-peaked distribution in Figure 4. First, as w/P falls, the risk premium rises. This in turn means that the specialist, who is holding a levered position in the risky asset, increases his wealth on average. This force is stronger as the risk

premium rises, which is why the distribution places almost no weight on risk premia as high as 30%. At the other end, when w/P is large so that w^h is small, the households are poor and consuming little but still receive labor income. Thus, their wealth grows as they save the labor income, which shifts the wealth distribution back towards the constrained region.

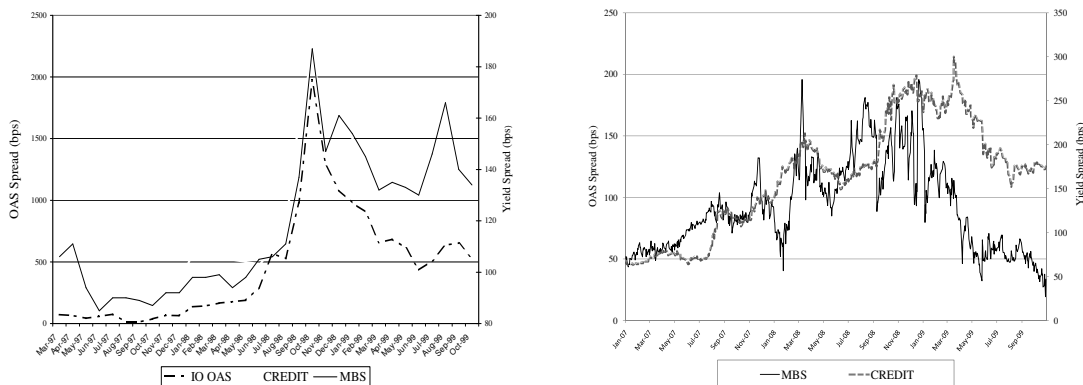
Table 3: Measurements

Panel A: Constrained and Unconstrained Regions								
We present average measurements for the economy, broken down into conditional on being in the constrained region, conditional on being in the unconstrained region, and unconditional average. Parameters are as given in Table 2. We also include a case for $m = 8$.								
	$m = 4$ Case			$m = 8$ Case				
	Avg.	Unc.	Const.	Avg.	Unc.	Const.		
Probability		64.45	33.55		80.39	19.61		
Risk Premium (%)	3.41	3.14	3.99	3.43	3.28	4.07		
Sharpe Ratio (%)	36.95	33.79	43.19	37.34	35.57	44.64		
Interest Rate (%)	0.62	0.87	0.12	0.58	0.72	0.00		
Debt/Assets Ratio (%)	55.29	50.26	65.24	54.86	51.89	67.08		
Income Ratio (%)	33.72	26.80	38.09	37.35	31.42	38.87		

Panel B: Measures at Different Risk Premia								
The second row reports the probability that the economy will ever reach a value of risk premium greater than the given π . The rest of the rows report measures at the given π .								
	$m = 4$				$m = 8$			
	3%	6%	9%	12%	3%	6%	9%	12%
Risk Premium (%) $\equiv \pi$								
Prob (Risk Premium $> \pi$)	93.93	1.58	0.26	0.08	94.55	1.01	0.17	0.06
Sharpe Ratio at π	31.89	65.46	101.46	140.67	31.88	65.95	100.89	136.95
Interest Rate at π	0.96	-1.77	-4.79	-8.05	0.95	-1.86	-4.89	-7.99
Debt/Assets Ratio at π	44.00	81.96	89.76	93.19	43.30	82.26	89.58	92.76

Table 3 provides further information on the range of variation of the state variable. Focusing on Panel A ($m = 4$ case), the economy spends 66.45% of the time in the unconstrained region. We may think of the unconstrained region as a “normal” non-crisis period. The average risk premium and Sharpe ratio, conditional on being in the unconstrained region, is 3.14% and 33.89, respectively (Panel A). In the constrained region, the risk premium rises to average 3.99%. The probability that the risk premium will exceed 6% is 1.58%. For the risk premium to exceed 9%, which is about triple the unconstrained region average in terms of both risk premium and Sharpe ratio, the probability is 0.26% (Panel B). An extreme crisis that increases risk premia and Sharpe ratio about 4X to 12% is very unlikely, in keeping with the historical record. Our model puts this probability at 0.08%.

Figure 5: Crisis Spreads



The left panel graphs the spreads between the Moody’s index of AAA corporate bonds and the 10 year Treasury rate (grey line, “credit”), the spreads between FNMA 6% TBA mortgage-backed securities and the 10 year Treasury rate (black line, “MBS”), and the option-adjusted spreads on a portfolio of interest-only mortgage-backed securities relative to Treasury bonds (dashed line, “IO OAS”) from 1997 to 1999. The right panel graphs the same credit spread as well as the OAS on the FNMA 6% MBS from 2007 to 2009.

To put the numbers from Table 3 in perspective, consider the 1998 crisis and the 2008 subprime crisis. Figure 5, left-panel graphs the behavior of the high grade credit spread (AAA bonds minus Treasuries), the spread on FNMA mortgage backed securities relative to Treasuries, and the option adjusted spread on volatile interest-only mortgage derivative securities (data are from Gabaix, Krishnamurthy, and Vigneron, 2007). The spreads are graphed over a period from 1997 to 1999 and includes the fall of 1998 hedge fund crisis. During 1997 and upto the middle of 1998 spreads move in a fairly narrow range. If we interpret the unconstrained states of our model as this “normal” period, then the muted response of risk premia to the state can capture this pre-crisis period. In a short period around October 1998 spreads on these securities increase sharply. The credit spreads and MBS spreads double from their pre-crisis level. The mortgage derivative spread increases by many multiples. The right-panel graphs the credit spread and the FNMA mortgage spread from 2007 to 2009. The subprime crisis begins in the summer of 2007, escalating until the fall of 2008. From the pre-crisis period to the fall of 2008, the MBS spread quadruples, while the credit spread rises six-fold. It is hard to say precisely how much Sharpe ratios increase during these episodes, because the underlying default risk in these bonds is also increasing. However,

a doubling or tripling is plausibly within the range of estimates. Our simulations suggest that the probability of the risk premium tripling from a normal level is 0.26%, indicating these crises are rare events. Moreover, from the standpoint of standard representative household asset pricing models, even a modest increase in risk premia during the 1998 event is difficult to understand as aggregate consumption was barely at risk. In our model, the asymmetry in the intermediation constraint calibrated to hedge fund data can generate the dramatic increase in risk premia around crises.

The table also presents an $m = 8$ case to gauge the sensitivity of the calibration to the choice of m , which is perhaps the hardest parameter to confidently pin down. The larger m leads to a smaller constrained region (“constraints” effect). The probability of falling into the constrained region is 19.61% for this case, compared to 33.54% for the $m = 4$ case. As discussed in Section 3.4, when m is larger the specialist is able to raise more external capital based on any given level of his own wealth. Thus his wealth has to be lower in order to fall into the constrained region. On the other hand, the risk premium in the constrained and unconstrained regions are higher in the $m = 8$ case. More generally, the higher m case also displays a “sensitivity” effect. When m is higher a \$1 fall in specialist wealth leads to an \$ m reduction in household contributions to the intermediary, creating a sharper rise of the risk premium for any given specialist wealth. The two effects of changing m roughly cancel out: risk premia are conditionally higher in the $m = 8$ case, but the economy is also less likely to fall into the constrained region.

5.4 Flight to quality

The row in Table 3, Panel A corresponding to the interest rate shows that the interest rate falls from an average of 0.62% in the unconstrained region to 0.12% in the constrained region. There are two intuitions behind this fall in interest rates. First, as the specialist’s consumption volatility rises with the tightness of the intermediation constraint, the precautionary savings effect increases specialist demand for the riskless bond. Second, as specialist wealth falls, households withdraw equity from intermediaries, increasing their demand for the riskless bond. To clear the bond market, the equilibrium interest rate has to fall. Both the behavior of the interest rate and the disintermediation-driven demand for bonds is consistent with a flight to quality.

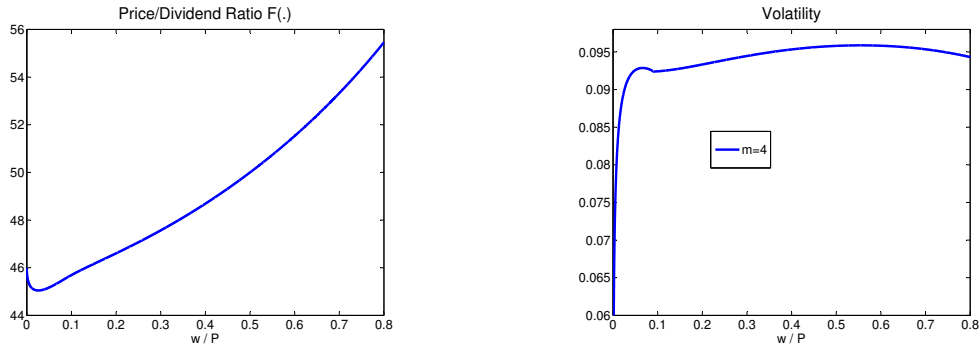
However, we can also see from the table that the interest rate is over-sensitive to the state in our model. At the 6% risk premium state, the interest rate is around -1.77% , falling to

−4.79% at the 9% risk premium state (for the $m = 4$ case).

The main reason for this over-sensitive interest rate is that we are pushing the general equilibrium of our model too far. Our model-economy consists of only an intermediation sector and therefore ascribes all movements in interest rates to shocks within that sector. In practice, part of the demand for bonds in the economy is from sectors that are unaffected by the intermediation constraint, so that it is likely that our model overstates the interest rate effect. However, it also does not seem appropriate to fix the interest rate exogenously, since interest rates do fall during a crisis episode. Thus, while the qualitative prediction of our model for interest rates seems correct, the quantitative implications are the least credible results of our analysis.

5.5 Price/Dividend Ratio and Volatility

Figure 6: P/D Ratio and Volatility



Price/Dividend ratio (left panel) and risky asset return volatility (right panel) are graphed against the specialist wealth as a percentage of the assets held by the intermediation sector (w/P). Parameters are given in Table 2.

The left-hand panel of Figure 6 graphs the price/dividend ratio $F(\cdot)$ against w/P . Consistent with intuition, over most of the range, $F(\cdot)$ falls as specialist wealth falls. There is a non-monotonicity that arises when the specialist wealth is very small – although this occurs for values of w/P for which the steady state distribution places very little weight (see Figure 4). The non-monotonicity arises because interest rates diverge to negative infinity when the specialist wealth approaches zero. There are two forces affecting the discount rates applied to

dividends in determining $F(\cdot)$: On the one hand, the risk premium is high when the specialist wealth is low; on the other hand, the interest rate is low for higher specialist wealth. These two effects combine to produce the non-monotonicity of $F(\cdot)$.

The right-hand panel of Figure 6 gives the pattern of the risky asset return volatility when the specialist wealth varies. Over most of the relevant range of variation of the state variable, the volatility is constant between 9% and 9.5%. In particular, the model fails to replicate the observed increase in conditional volatility accompanying a crisis period.

The non-monotonicity in $F(\cdot)$ also causes volatility to fall in the region where w/P approaches zero. The risky asset price is equal to $D_t \times F(w_t/P_t)$. The non-monotonicity means that a shock that causes a fall in D_t leads to a rise in F (since w_t/P_t decreases as D_t falls, and $F'(\cdot) < 0$). We stress again that the steady state distribution places almost no weight on these small values of w/P .

5.6 Capital movement and recovery from crisis

Referring to Figure 5, left-panel, the corporate bond spread and MBS spread widen from 90 bps in July 1998 to a high of 180 bps in October 1998 before coming down to 130 bps in June 1999. Thus, the half-life — that is, the time it takes the spread to fall halfway to the pre-crisis level — is about 10 months. The interest-only mortgage derivative spread, which is very sensitive to market conditions, widens from 250 bps in July 1998 to a high of 2000 bps before coming back to 500 bps in June 1999. In the right-panel, the MBS spread recovers back to its pre-crisis level by June 2009, while the credit spread remains elevated through the end of the period. We note that this timescale for mean reversion, on the order of months, is much slower than the daily mean-reversion patterns commonly addressed in the market micro-structure literature (e.g., Campbell, Grossman, and Wang, 1994).

A common wisdom among many observers is that this pattern of recovery reflects the slow movement of capital into the affected markets (Froot and O’Connell, 1999, Berndt, et. al., 2004, Mitchell, Pedersen, and Pulvino, 2007, Duffie and Strulovici, 2009). Our model captures this slow movement. We will show in this section that our baseline calibration can also replicate these speeds of capital movement.

In the crisis states of our model, risk premia are high and the specialists hold leveraged positions on the risky asset. Over time, profits from this position increase w_t , thereby increasing the capital base of the intermediaries. The increase in specialist capital is mirrored by an m -fold increase in the allocation of households’ capital to the intermediaries, as the

intermediation constraint is relaxed. Together these forces reflect a movement of capital back into the risky asset market and lead to increased risk-bearing capacity and lower risk premia. Note, however, that one dimension of capital movement that plausibly occurs in practice but is not captured by our model is the entry of “new” specialists into the risky asset market.

We can use the model simulation to gauge the length and severity of a crisis within our model. Table 4 presents data on how long it takes to recover from a crisis in our model. We fix a state (y, D) corresponding to an instantaneous risk premium in the “Transit from” row. Simulating the model from that initial condition, we compute and report the first passage time that the state hits the risk premium corresponding to the “Transit to” column. The time is reported in years.

Table 4: Crisis Recovery

This table presents transition time data from simulating the model. We fix a state corresponding to an instantaneous risk premium of 12% (“Transit from”). Simulating the model from that initial condition, we compute and report the first passage time that the state hits the risk premium corresponding to that in the “Transit to” column. Time is reported in years. The column “Increment time” reports the time between incremental “Transit to” rows.						
Transit to	10	7.5	6	5	4	3.5
Transit time from 12	0.17	0.66	1.49	2.72	5.88	9.84
Increment time	0.17	0.49	0.83	1.24	3.15	3.97

If we start from the extreme crisis state of 12% and compute how long it takes to recover to 7.5% — i.e. halfway back to the unconditional average levels we report earlier of around 3% — the time is 0.66 years (7.9 months). From the 10% crisis state to the 6.5% state (halfway to 3%) takes 0.93 years (this number is not reported in the table). For the fall of 1998 episode, the half-life we suggested was around 10 months. The model half-life from 10% is of the same order of magnitude of the empirical observation.

The slow adjustment of risk premia, in timescales of many months, during the 1998 episode is also consistent with other studies of crisis episodes. Berndt, et. al. (2005) study the credit default swap market from 2000 to 2004 and note a dramatic market-wide increase in risk premia (roughly a quadrupling) in July 2002 (see Figures 1 and 2 of the paper). Risk premia gradually fall over the next two years: From the peak in July 2002, risk premia halve by April 2003 (9 months). The authors argue that dislocations beginning with the Enron crisis led to a decrease in risk-bearing capacity among corporate bond investors. Mirroring the decreasing risk-bearing capacity, risk premia rose before slowly falling as capital moved back

into the corporate bond market and expanded risk bearing capacity. Gabaix, Krishnamurthy, and Vigneron (2007) note a dislocation in the mortgage-backed securities in late 1993 triggered by an unexpected wave of consumer prepayments. A number of important hedge fund players suffered losses and went out of business during this period, leading to a reduction in risk bearing capacity. Figure 3 in the paper documents that risk premia reached a peak in December 1993 before halving by April 1994 (5 months). Froot and O’Connell (1999) study the catastrophe insurance market and demonstrate similar phenomena. When insurers suffer losses that deplete capital they raise the price of catastrophe insurance. Prices then gradually fall back to long-run levels as capital moves back into the catastrophe insurance market. Froot and O’Connell show that the half-life in terms of prices can be well over a year.¹²

Each of these markets are intermediated markets that fit our model well. Investors are institutions who have specialized expertise in assessing risk in their markets. Our theory explains the slow movement of risk bearing capacity and risk premia documented in these case-studies. The calibrated model also captures the frequency of the slow adjustment of risk premia.

6 Crisis Policy Experiments

We study the effect of policy interventions in the crisis of the model. We study three policies: (1) Lowering borrowing rates to the intermediary, as with a decrease in the central bank’s discount rate; (2) Direct purchase of the risky asset by the government, financed by debt issuance and taxation of households; and, (3) Infusing equity capital into the intermediaries during a crisis. These three policies are chosen because they are among those undertaken by central banks in practice. Our aim is to quantify the effects of these policies on the equilibrium of our model. The analysis is purely positive, and we make no claims as to optimality.

Our policy experiments correspond to the following exercise. Suppose we are in a crisis state currently, with a given asset/liability position for the households and specialists. From this initial condition, suppose that the government conducts a policy that was not anticipated by the agents. We trace the effects of this policy on the recovery of the economy from that

¹²Mitchell, Pedersen, and Pulvino (2007) document similar effects in the convertible bond market in 1998 and again in 2005. In both cases, crisis recovery times are in the order of months. They also note that spreads in merger arbitrage strategies took several months to recover following the October 1987 risky asset-market crash.

crisis state.¹³

To be more precise, we compute two equilibria, one with the policy and one without the policy. For example, the first policy we consider is a borrowing subsidy that is given to intermediaries as long as the economy is in the constrained region. We write the subsidy as a function of the primitive state variables and solve the equilibrium of the model under such a policy. We then suppose that the economy is currently in a given crisis state of the no-policy equilibrium (12% risk premium state in the simulations), characterized by the portfolio positions of the households and the specialists. The government policy enacted in this state causes asset prices to jump because the policy is unanticipated. From that point on, the dynamics of the economy are described by the solution to our model under the with-policy equilibrium.¹⁴

6.1 Borrowing Subsidy

During financial crises, the central bank lowers its discount rate and its target for the overnight interbank interest rate. Financial intermediaries rely heavily on rolling over one-day loans for their operation (see, for example, Adrian and Shin (2010) on the overnight repurchase market). Because of this dependence, intermediaries are perhaps the most sensitive sector within the economy to overnight interest rates. Commercial and investment banks have access to overnight funds at the discount window of the central bank. Thus, to the extent that the central bank lowers overnight rates, including the discount rate, it reduces the borrowing costs of financial intermediaries.

While our model does not have a monetary side within which to analyze how a central bank alters the equilibrium overnight interest rate, we can go some way towards examining the effect of this policy by studying the following transfer. The debt position of intermediaries at date t is $(\alpha_t^I - 1)w_t$. Suppose that the government makes a lumpsum transfer of $\Delta r \times (\alpha_t^I - 1)w_t dt$ from households to intermediaries, where Δr measures the size of the transfer. The transfer

¹³The government policy is a zero-probability event in our exercise. Another experiment would be to study a policy that is expected to be enacted given some value of the state variable – say the government infuses equity capital if the risk premium touches 12%. Such a policy would be anticipated by agents within the equilibrium of the model. Analyzing such a policy does not pose any difficulty for our modeling structure, but it adds an extra layer of complexity to the model. For the sake of brevity, we have opted to focus on the simpler experiment.

¹⁴The initial condition from which we simulate the with-policy equilibrium is chosen so that it matches the portfolio holdings of the household in the 12% risk premium state of the no-policy equilibrium.

is proportional to the debt of the intermediary.

The subsidy experiment can be thought of as a reduction in the central bank’s discount rate. In practice, when the central bank makes funds available more cheaply to the financial sector through the discount window it is transferring real resources from taxpaying households to the financial sector. However, since our model is cast in real terms, the subsidy is only a stand-in for a reduction in something like the overnight Federal Funds rate.

Formally, we examine an equilibrium where Δr is paid only if $w > w^c$. For $w < w^c$ there is no subsidy. We express this transfer of $\Delta r \times (\alpha_t^I - 1)w_t dt$ in terms of the primitive state variables y_t and D_t . Then, the dynamic budget constraints of household and specialist are altered to account for the transfer (see equation 10), and this change is traced through to rederive the ODE for the price/dividend ratio (see Appendix C for details).

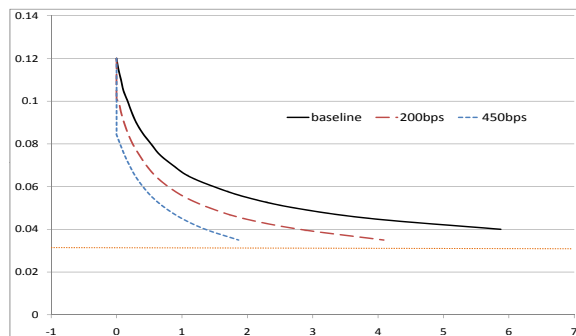
Table 5: Borrowing Subsidy

This table presents transition time data from simulating the model. We begin in the 12% risk premium state and report the first passage time for the state to reach that in the first column of the table (“Transit to” column). Time is reported in years. We report the case of no subsidy ($\Delta r = 0$), as well as subsidies of 0.01, 0.02 and 0.045. A subsidy of 0.01 corresponds to 100 <i>bps</i> . The first row of the table reports the instantaneous jump downwards in the risk premium when the government initiates the policy.				
Transit to	$\Delta r = 0$	$\Delta r = 0.01$	$\Delta r = 0.02$	$\Delta r = 0.045$
		11.05%	10.11%	8.38%
10	0.17	0.10	0.03	
7.50	0.66	0.48	0.34	0.12
6	1.49	1.08	0.79	0.40
5	2.72	1.94	1.43	0.75
4	5.88	4.02	2.81	1.36

Table 5 presents the results. We start the economy in the state corresponding to the 12% risk premium. The subsidy of Δr is provided to the intermediaries as long as the economy is in the constrained region. The table reports the recovery times from the 12% extreme crisis state for different levels of Δr . Consistent with intuition, a higher subsidy speeds up the recovery process. The 200 *bps* subsidy speeds up the recovery to 7.5% by 0.32 years. Note that from August 2007 to October 2008, the discount rate decreased by 450 *bps*. The last column in the table indicates the effect of this policy within our model. Figure 7 presents the

data graphically.

Figure 7: Response to borrowing subsidy



The figure describes the path of recovery, measured in terms of risk premium, to a shock that moves the economy at $t = 0$ to the 12% risk premium state. The recovery path is drawn for different levels of borrowing subsidy given to intermediaries. The horizontal line indicates the unconstrained average risk premium of 3.14%.

6.2 Direct Asset Purchase

In both the subprime crisis as well as the Great Depression the government directly entered the asset market to purchase distressed assets. The Federal Reserve and GSEs purchased nearly \$1.8tn of mortgage-backed securities over the period from August 2007 to August 2009 (\$1.25tn by the Federal Reserve and \$550bn by the GSEs). We can evaluate the impact of this policy as follows. Suppose that the government purchases a fraction s the risky asset in states $w < w^c$, financing this purchase by issuing sP of instantaneous debt (where P is the price of the risky asset). The cash-flow, after repaying debt, from this transaction is $sP(dR_t - r_t dt)$. We assume that the government raises lumpsum taxes from (or rebates to) the households to balance this cash-flow.

Table 5 reports the results for three values of s , which is the share of the intermediated risky asset market that the government purchases. If we take the stock of intermediated assets to be \$15tn, then the \$1.8tn number cited above is 12% of this stock. We assume that the

policy is initiated in the state corresponding to 12% risk premium and not removed until the economy is in the unconstrained region. We trace the recovery path from this state.

Table 6: Asset Purchase

This table presents transition time data from simulating the model. We begin in the 12% risk premium state and report the first passage time for the state to reach that in the first column of the table (“Transit to” column). Time is reported in years. We report the case of no purchase ($s = 0$), as well as purchases of 0.04, 0.08, and 0.12. A purchase with $s = 0.04$ corresponds to the government buying 4% of the outstanding stock of intermediated risky assets. The first row of the table reports the instantaneous jump downwards in the risk premium when the government begins its purchase.

Transit to	$s = 0$	$s = 0.04$	$s = 0.08$	$s = 0.12$
		11.63%	11.18%	10.77%
10	0.17	0.17	0.14	0.10
7.50	0.66	0.68	0.63	0.58
6	1.49	1.47	1.44	1.40
5	2.72	2.70	2.64	2.56
4	5.88	5.87	5.81	5.63

The policy causes a downward jump in the risk premium. The asset purchase policy indirectly increases the household’s exposure to the risky asset because future taxes now depend on the returns to the risky asset. In turn, this means that specialists bear less risk in equilibrium and hence the risk premium falls. Effectively this policy puts less risk on the limited risk-bearing capacity of the intermediary sector. After this initial jump the recovery path is almost the same as the case of no intervention. For example, if we compare the incremental time it takes the economy to move from 7.5% to 6%, we see that the time for the no intervention case is 0.83 years, while it is 0.82 years for the case of $s = 0.12$. Intuitively the purchase has no further effect because there is a countervailing force: the specialist holds a smaller position in the risky asset (since the taxpayer holds a larger share) and hence less of the risk premium accrues to it, which causes intermediary capital to recover more slowly.

6.3 Capital infusion

A number of crisis interventions are aimed at increasing the equity capital of intermediaries. For example, in the Great Depression, the government directly acquired preferred shares in

banks, thereby increasing their equity capital. In the subprime crisis, the U.S. Treasury purchased \$205 bn of preferred shares in the intermediary sector through the capital purchase program.

We examine an equilibrium in which m is increased to $\bar{m} > 4$ in a crisis, defined as states where $w < w^*$, or equivalently $y > y^*$. The higher m indicates that the intermediary increases its equity capital proportionate to $\bar{m} - m$. The extra equity capital is purchased by the government, and paid for by lumpsum taxes on the households. Returns on the government investment are rebated in a lumpsum fashion to the households. We think of the increase in m as a temporary relaxation of the equity capital constraint. For example, one may imagine that the government is temporarily able to monitor intermediaries better than households during a crisis and can thus relax the capital constraint. Our aim is to quantify the effect of the relaxation of the constraint on the crisis recovery.

To evaluate the Treasury's policy with our model, we need to choose \bar{m} and w^* . As with the asset purchase policy, we express the policy formally in terms of y . We set y^* equal to y^c , so that the policy is reversed when the economy enters the unconstrained region. For technical reasons, to avoid a discontinuity in m , we increase m from m to \bar{m} over an interval from y^* to $y^* + 0.22$, where 0.22 is the drift of y around y^* in the new equilibrium. Our aim here is to implement the policy to last 1 year from the time at which it is initiated, matching the duration of the stimulus we have observed in practice. Our results are not sensitive to the choice of 0.22.

We choose \bar{m} to represent the Treasury's purchase of \$205 bn of bank capital. Note that capital in our model refers to common shares, while in practice, the Treasury purchased preferred shares. The distinction is important because our model works through the sharing of risk between the specialist and the household/government, rather than directly through the amount of funds that are transferred to the intermediary sector. When the government invests in the intermediary and shares some of the risk in the specialist's investment, then the specialist bears less risk in equilibrium and the risk premium adjusts downwards. The returns on common shares are more sensitive to the returns on intermediary investment than are the returns on preferred shares, indicating that common shares allow for more risk sharing than preferred shares. Franks and Torous (1994), based on a sample of distressed firms over the period 1983 to 1988, document that in a bankruptcy/reorganization, preferred shares are repaid 42% of face value. In our model, the value of common shares approaches zero as the value of assets falls towards the value of liabilities (the bankruptcy threshold). Likewise, as the value of assets rises, preferred shares received a relatively fixed dividend, while the value

of common shares increases. We translate the preferred share purchase in terms of common shares using the 42% number of Franks and Torous. We assume that an injection of \$1 of preferred shares is equal to an injection of \$0.58 ($= 1 - 0.42$) of common shares.

The Treasury’s capital injection was distributed across many banks, from pure lending institutions to trading institutions. Since our model is primarily about securities markets and trading institutions, we apportion the \$205 bn capital to reflect the injection of capital to support securities trading. We multiply the injection by 0.40, which is the fraction of securities in total bank assets, as computed from the Flow of Funds 2007 data. We thus evaluate the effect in our model of raising m to \bar{m} in the crisis states, where \bar{m} is chosen so that the implied increase in equity capital (as fraction of total assets under intermediation) in the 12% crisis state is \$48 bn ($=205 \times 0.58 \times 0.40$) divided by \$15 tn. We also present results for a \$38 bn and \$58 bn equity injection. The results are in Table 7.

Table 7: Equity Injection

This table presents transition time data from simulating the model. We begin in the 12% risk premium state and report the first passage time for the state to reach that in the first column of the table (“Transit to” column). Time is reported in years. We report the case of a purchase of equity capital of \$38 bn, \$48 bn and \$58 bn, which is reversed in roughly one year. The first row of the table reports the instantaneous jump downwards in the risk premium when the government injects the equity capital.				
Transit to	Baseline	\$38bn	\$48bn	\$58bn
		9.12%	8.67%	8.28%
7.50	1.09	0.39	0.31	0.23
6	1.49	1.14	1.05	0.96
5	2.72	2.33	2.20	2.10
4	5.88	5.13	5.06	4.98

The effects of policy are qualitatively similar to the other cases: there is a jump downwards in the risk premium and a gradual adjustment afterwards. It is most interesting to compare the effects of the three policies. Compared to the asset purchase case, we see that a relatively small amount of funds used towards equity purchase produces a much faster recovery. The reason that the equity injection has such a large effect is because the fundamental friction in our model is an equity capital constraint. The equity capital injection of \$48 bn, corresponding to the actually policy enacted in 2008/2008, leads to a recovery time to the 6% state of 1.05 years. The 450 basis point borrowing subsidy, corresponding to actual policy, leads to the

fastest recovery time of 0.40 years. This policy has a large effect because the intermediaries are very leveraged in our model, carrying a debt/asset ratio of 82% in the 12% risk premium state. These numbers concern the benefits of these policies. To provide a more comprehensive assessment of the policies, one also needs to evaluate the costs of these interventions.

7 Conclusion

We have presented a model to study the dynamics of risk premia in a crisis episode where intermediaries' equity capital is scarce. We calibrate the model and show the model does well in matching two aspects of crises: the nonlinearity of risk premia in crisis episodes; and, the recovery from crises in the order of many months. We also use the model to evaluate the effectiveness of central bank policies, finding that infusing equity capital into intermediaries is the most effective policy in our model.

A limitation of our model is that it does not shed any light on the connection between the performance of intermediated asset markets we model (i.e. the mortgage-backed securities market) and the aggregate stock market. Yet, as we have seen during the subprime crisis, the deterioration in intermediation does spillover to the S&P500. It will be interesting to explore such a connection by introducing a second asset, in positive supply, that the households invest in directly. Such an asset can represent the S&P500 and may shed light on the equity premium puzzle. Introducing such an asset is also likely to dampen the over-sensitive interest rate effect that is present in our model.

References

- [1] Adrian, Tobias and Shin, Hyun Song, 2010, “Liquidity and Leverage,” *Journal of Financial Intermediation*, forthcoming.
- [2] Allen, Franklin and Douglas Gale, 1994, Limited Market Participation and Volatility of Asset Prices, *American Economic Review* 84, 933-955.
- [3] Allen, Franklin and Douglas Gale, 2005, “From Cash-in-the-Market Pricing to Financial Fragility,” *Journal of the European Economic Association*, 3(2-3), 535-546.
- [4] Alvarez, Fernando, Andrew Atkeson, and Patrick Kehoe, 2002, “Money, interest rates, and exchange rates with endogenously segmented markets,” *Journal of Political Economy* 110, 73—112.
- [5] Barberis, N., M. Huang, and T. Santos, 2001, Prospect theory and asset price, *The Quarterly Journal of Economics*, 116, 1-53.
- [6] Basak, Suleyman and Domenico Cuoco, 1998, An Equilibrium Model with Restricted risky asset Market Participation, *Review of Financial Studies*, 11, 309-341.
- [7] Bates, David, 2003, Empirical Option Pricing: A Retrospection, *Journal of Econometrics*, 116, 387–404.
- [8] Berndt, Antje, Rohan Douglas, Darrell Duffie, Mark Ferguson, and David Schranz, 2004, Measuring Default Risk Premia from Default Swap Rates and EDFs, mimeo, Stanford University.
- [9] Borodin, A.N. and P. Salminen, 1996, *Handbook of Brownian Motion*. Birkhauser, Boston, MA.
- [10] Boudoukh, Jacob, Matthew Richardson, Richard Stanton, and Robert Whitelaw, 1997, Pricing mortgage-backed securities in a multifactor interest rate environment: A multivariate density estimation approach, *Review of Financial Studies* 10, 405-446.
- [11] Brunnermeier, Markus and Lasse Pedersen, 2009, Market Liquidity and Funding Liquidity, *Review of Financial Studies*, 22(6), 2201-2238.
- [12] Brunnermeier, Markus and Yuliy Sannikov, 2010, “A Macroeconomic Model with a Financial Sector,” working paper, Princeton University.

- [13] Campbell, John and John Cochrane, 1999, “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior,” *Journal of Political Economy*, 107, 205-251.
- [14] Campbell, John, Sanford Grossman and Jiang Wang, 1993, “Trading Volume and Serial Correlation in risky asset Returns,” *Quarterly Journal of Economics*, 108, 905-940.
- [15] Collin-Dufresne, Pierre, Robert Goldstein, and J. Spencer Martin, 2001, The Determinants of Credit Spread Changes, *The Journal of Finance*, 66, 2177-2207.
- [16] Diamond, Douglas, 1997, “Liquidity, Banks, and Markets,” *Journal of Political Economy*, 105(5), pp. 928-56.
- [17] Diamond, Douglas and Philip Dybvig, 1983, “Bank Runs, Deposit Insurance, and Liquidity,” *Journal of Political Economy*, 91(5), pp. 401-19.
- [18] Diamond, Douglas and Raghuram Rajan, 2005, “Liquidity Shortages and Banking Crises,” *Journal of Finance* 60 (2), pp. 615-647.
- [19] Duffie, Darrell and Bruno Strulovici, 2009, “Capital Mobility and Asset Pricing,” working paper, Stanford University.
- [20] Dumas, Bernard, 1989, “Two-Person Dynamic Equilibrium in the Capital Market,” *Review of Financial Studies*, 2(2), 157–188.
- [21] Edmond, Chris and Pierre-Olivier Weill, 2009, “Aggregate Implications of Micro Asset Market Segmentation,” NBER Working Paper 15254.
- [22] Franks, Julian and Walter Torous, 1994, “A Comparison of Financial Recontracting in Distressed Exchanges and Chapter 11 Reorganizations,” *Journal of Financial Economics*, 35, 349-370.
- [23] Froot, Kenneth A., and Paul G. O’Connell, 1999, The Pricing of US Catastrophe Reinsurance in *The Financing of Catastrophe Risk*, edited by Kenneth Froot, University of Chicago Press.
- [24] Fung, William, and David Hsieh, 2006, “Hedge Funds: An Industry in Its Adolescence,” *Federal Reserve Bank of Atlanta Economic Review*, (Fourth Quarter), 91, 1-33

- [25] Gabaix, Xavier, Arvind Krishnamurthy and Olivier Vigneron, 2007, Limits of Arbitrage: Theory and Evidence from the MBS market, *Journal of Finance*, 62(2), 557-596.
- [26] Garleanu, Nicolae, Lasse Heje Pedersen, and Allen M. Poteshman, 2005, Demand-Based Option Pricing, Wharton mimeo.
- [27] Gatev, Evan, and Phillip Strahan, 2006, Banks Advantage in Supplying Liquidity: Theory and Evidence from the Commercial Paper Market, *Journal of Finance*, Vol. 61, No. 2, pp. 867-92.
- [28] Geanakoplos, John, and Ana Fostel, 2008, "Leverage Cycles and the Anxious Economy," *American Economic Review*, Vol. 98, No. 4, pp. 1211-244.
- [29] Gromb, Denis and Dimitri Vayanos, 2002, Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs, *Journal of Financial Economics* 66, 361-407.
- [30] He, Zhiguo, and Arvind Krishnamurthy, 2009, "A Model of Capital and Crisis," NBER working paper #14366.
- [31] He, Zhiguo, InGu Khang and Arvind Krishnamurthy, 2010, "Balance Sheet Adjustments during the 2008 Crisis," forthcoming, *IMF Economic Review*.
- [32] Holmstrom, Bengt and Jean Tirole, 1997, Financial Intermediation, Loanable Funds, and the Real Sector, *Quarterly Journal of Economics* 112, 663-691.
- [33] Huang, Jennifer and Jiang Wang, 2007, "Liquidity and Market Crashes," forthcoming, *Review of Financial Studies*.
- [34] Karatzas, I. and S. Shreve, 1998, *Methods of Mathematical Finance*. New York: Springer-Verlag.
- [35] Kiyotaki, N. and J. Moore, 1997, Credit Cycles, *Journal of Political Economy*, 105 (2), 211-248.
- [36] Kyle, Albert and Wei Xiong, 2001, "Contagion as a Wealth Effect," *Journal of Finance* 56, 1401-1440.
- [37] Lucas, Robert E., Jr, 1978, Asset Prices in an Exchange Economy, *Econometrica*, 46, 1429-1445,

- [38] Mankiw, Gregory N, and Stephen Zeldes, 1991, The Consumption of risky assetholders and Non-risky assetholders, *Journal of Financial Economics*, 29, pp. 97-112.
- [39] Mitchell, Mark, Lasse Pedersen, and Todd Pulvino, 2007, "Slow Moving Capital," *American Economic Review Papers and Proceedings*, forthcoming.
- [40] Pennacchi, George, 2006, "Deposit insurance, bank regulation and financial system risk," *Journal of Monetary Economics*, Vol. 53, No. 1, pp. 1-30.
- [41] Shleifer, Andrei and Robert W. Vishny, 1997, The Limits of Arbitrage, *Journal of Finance* 36, 35-55.
- [42] Vayanos, Dimitri, 2004, Flight to Quality, Flight to Liquidity, and the Pricing of Risk, working paper, LSE.
- [43] Vissing-Jorgensen, Annette, 2002, Limited asset market participation and the elasticity of intertemporal substitution, *Journal of Political Economy*, 110(4), 825-853.
- [44] Wang, Jiang, 1996, "The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogeneous Investors," *Journal of Financial Economics*, 41(1), 75-110.
- [45] Warther, Vincent, 1995, Aggregate mutual fund flows and security returns, *Journal of Financial Economics*, 39, 209-235.
- [46] Xiong, Wei, 2001, Convergence Trading with Wealth Effects: An Amplification Mechanism in Financial Markets, *Journal of Financial Economics* 62, 247-292

A ODE Solution

In this appendix, we detail the ODE that characterizes the equilibrium. We analyze our ODE based on state variable y , i.e., the scaled households wealth. Denote the dynamics of y_t as,

$$dy_t = \mu_y dt + \sigma_y dZ_t, \quad (21)$$

for unknown functions μ_y and σ_y

We write $\frac{dc_t}{c_t}$ and dR_t as functions of μ_y, σ_y and the derivatives of $F(y)$. Because $c_t = D_t(1 + l - \rho y_t)$, we have

$$\begin{aligned} \frac{dc_t}{c_t} &= \frac{dD_t}{D_t} - \frac{\rho dy}{1 + l - \rho y} - \frac{\rho}{1 + l - \rho y} Cov_t \left[dy, \frac{dD}{D} \right] \\ &= \left(g - \frac{\rho}{1 + l - \rho y} (\mu_y + \sigma_y \sigma) \right) dt + \left(\sigma - \frac{\rho \sigma_y}{1 + l - \rho y} \right) dZ_t. \end{aligned}$$

We also have

$$dR_t = \frac{dP_t + D_t dt}{P_t} = \left[g + \frac{F'}{F} \mu_y + \frac{1}{2} \frac{F''}{F} \sigma_y^2 + \frac{1}{F} + \frac{F'}{F} \sigma_y \sigma \right] dt + \left(\sigma + \frac{F'}{F} \sigma_y \right) dZ_t.$$

Substituting these expressions into (16) we obtain the following ODE,

$$\begin{aligned} g + \frac{F'}{F} \mu_y + \frac{1}{2} \frac{F''}{F} \sigma_y^2 + \frac{1}{F} + \frac{F'}{F} \sigma_y \sigma &= \rho + \gamma g - \frac{\gamma \rho^h}{1 - \rho^h y} (\mu_y + \sigma_y \sigma) \\ &+ \gamma \left(\sigma - \frac{\rho^h}{1 - \rho^h y} \sigma_y \right) \left(\sigma + \frac{F'}{F} \sigma_y \right) - \frac{1}{2} \gamma (\gamma + 1) \left(\sigma - \frac{\rho^h}{1 - \rho^h y} \sigma_y \right)^2. \end{aligned} \quad (22)$$

A.1 Derivation of μ_y and σ_y

We rewrite equation (10) which describes the wealth dynamics (budget constraint) of the household sector as:

$$dw^h = \theta_s dP + D \theta_s dt + r \hat{\theta}_b dt + l D_t dt - \rho w^h dt. \quad (23)$$

In this equation,

$$\theta_s = \alpha^I \alpha^h (1 - \lambda) \frac{w^h}{P} \quad (24)$$

are the number of shares that the risky asset household owns, and

$$\hat{\theta}_b D = w^h - \theta_s P \quad (25)$$

is the amount of funds that the risky asset and debt households together have invested in the riskless bond. α^h and α^I are defined in the text and depends on whether the economy is constrained or not.

We apply Ito's Lemma to $P = DF(y)$ to find expressions for the drift and diffusion of dP . We can then substitute back into equation (23) to find expressions for the drift and diffusion of dw^h .

Now, we have defined $w^h = Dy$. We apply Ito's Lemma to this equation to arrive at a second expression for the drift and diffusion of dw^h . Matching the drift and diffusion terms from these two ways of writing dw^h , we solve to find μ_y and σ_y .

The result of this algebra is that:

$$\sigma_y = -\frac{\hat{\theta}_b}{1 - \theta_s F'} \sigma,$$

and,

$$\mu_y = \frac{1}{1 - \theta_s F'} \left(\theta_s + l + (r + \sigma^2 - g) \hat{\theta}_b - \rho y + \frac{1}{2} \theta_s F'' \sigma_y^2 \right).$$

A.2 ODE

Substituting for μ_y derived in (A.1) into (22), we find,

$$\begin{aligned} & \left(\frac{F'}{F} + \frac{\gamma \rho}{1 + l - \rho y} \right) \left(\frac{1}{1 - \theta_s F'} \right) \left(\theta_s + l + \hat{\theta}_b (r - g) - \rho y \right) + \frac{1}{F} \\ & \quad + \frac{1}{2} F'' \sigma_y^2 \left(\frac{1}{1 - \theta_s F'} \right) \left(\frac{1}{F} + \theta_s \frac{\gamma \rho}{1 + l - \rho y} \right) \\ = & \rho + g(\gamma - 1) + \gamma \left(\sigma - \frac{\rho}{1 + l - \rho y} \sigma_y \right) \left(\sigma + \frac{F'}{F} \sigma_y \right) \\ & \quad - \frac{1}{2} \gamma(\gamma + 1) \left(\sigma - \frac{\rho}{1 + l - \rho y} \sigma_y \right)^2 \end{aligned}$$

where,

$$\begin{aligned} r = & \rho + g\gamma - \frac{\rho\gamma}{1 + l - \rho y} \frac{\theta_s + l + (r - g) \hat{\theta}_b - \rho y + \frac{\sigma^2}{2} \theta_s F'' \frac{\hat{\theta}_b^2}{(1 - \theta_s F')^2}}{1 - \theta_s F'} \\ & - \frac{\gamma(\gamma + 1)\sigma^2}{2} \left(1 + \frac{\rho \hat{\theta}_b}{1 + l - \rho y} \frac{1}{1 - \theta_s F'} \right)^2 \end{aligned}$$

We define a function, $G(y) \equiv \frac{1}{1 - \theta_s F'}$; with this definition, we can write $G' = \theta_s G^2 F''$, and

$$\sigma_y = -\frac{\hat{\theta}_b}{1 - \theta_s F'} \sigma = -\hat{\theta}_b \sigma G.$$

Therefore we have

$$\begin{aligned} G' \frac{(\hat{\theta}_b \sigma)^2}{2} G \left(\frac{1}{\theta_s F} + \frac{\gamma \rho^h}{1 + l - \rho y} \right) = & \rho + g(\gamma - 1) - \frac{1}{F} \\ & + \frac{1}{2} \gamma \sigma^2 \left(1 + \frac{\rho}{1 + l - \rho y} \hat{\theta}_b G \right) \left(\frac{2(y - G \hat{\theta}_b)}{\theta_s F} - (1 + \gamma) \frac{1 + l - \rho y + \rho G \hat{\theta}_b}{1 + l - \rho y} \right) \\ & - \left(\frac{G - 1}{\theta_s F} + \frac{\gamma \rho^h}{1 + l - \rho y} G \right) \left(\theta_s + l + \hat{\theta}_b (r - g) - \rho y \right) \end{aligned}$$

and

$$r = \frac{\rho + g\gamma - \frac{\rho\gamma G}{1+l-\rho y} \left(\theta_s + l - g\hat{\theta}_b - \rho y + \frac{\sigma^2}{2} G' \hat{\theta}_b^2 \right) - \frac{\gamma(\gamma+1)\sigma^2}{2} \left(1 + \frac{\rho\hat{\theta}_b G}{1+l-\rho y} \right)^2}{1 + \frac{\rho\gamma G \hat{\theta}_b}{1+l-\rho y}}.$$

We combine these two pieces, using the relation, $\hat{\theta}_b \left(\frac{G-1}{\theta_s F} + \frac{\gamma\rho^h G}{1+l-\rho y} \right) = -\frac{y-G\hat{\theta}_b}{\theta_s F} + \frac{1+l-\rho y + \gamma\rho^h G \hat{\theta}_b}{1+l-\rho y}$, and arrive at a final expression of the ODE:

$$\begin{aligned} & G' \frac{(\hat{\theta}_b \sigma)^2}{2} \frac{G}{\theta_s F} \left(\frac{1+l+\rho y(\gamma-1)}{1+l-\rho y + \rho\gamma G \hat{\theta}_b} \right) \\ &= \rho + g(\gamma-1) - \frac{1}{F} + \frac{\gamma(1-\gamma)\sigma^2}{2} \left(1 + \frac{\rho G \hat{\theta}_b}{1+l-\rho y} \right) \frac{y-G\hat{\theta}_b}{\theta_s F} \left[\frac{1+l-\rho y - \rho G \hat{\theta}_b}{1+l-\rho y + \rho\gamma G \hat{\theta}_b} \right] \\ & \quad - \left(\frac{(1+l-\rho y)(G-1)}{\theta_s F} + \gamma\rho^h G \right) \frac{\theta_s + l + \hat{\theta}_b(g(\gamma-1) + \rho) - \rho y}{1+l-\rho y + \rho\gamma G \hat{\theta}_b}. \end{aligned} \quad (26)$$

The expressions for the bond holding $\hat{\theta}_b$ and risky asset holding θ_s depend on whether the economy is constrained or not. In the unconstrained region, as shown in Section 3.3, $\alpha^h = 1$, and $\alpha^I = \frac{F}{F-\lambda y}$. Utilizing (25) and (24), we have $\theta_s = \frac{(1-\lambda)y}{F-\lambda y}$, and $\hat{\theta}_b = \lambda y \frac{F-y}{F-\lambda y}$. In the constrained region $\alpha^h = \frac{m(F-y)}{(1-\lambda)y}$, $\alpha^I = \frac{1}{1+m} \frac{F}{F-y}$, therefore $\theta_s = \frac{m}{1+m}$, and $\hat{\theta}_b = y - \frac{m}{1+m} F$. Finally, as illustrated in Section 3.3, the cutoff for the constraint satisfies $y^c = \frac{m}{1-\lambda+m} F(y^c)$, and the economy is in the unconstrained region if $0 < y \leq y^c$.

A.3 Boundary conditions and technical parameter restriction

The upper boundary condition is described in Section 3.5. A lower boundary condition occurs when $y \rightarrow 0$. This case corresponds to one where specialists hold the entire financial wealth of the economy. Using L'Hopital's rule, it is easy to check that $\frac{G-1}{\theta_s F} \rightarrow \frac{F'(0)}{F(0)}$. Plugging this result into (26), and noting that both θ_s and $\hat{\theta}_b$ go to zero as y goes to zero, we obtain,

$$F(0) = \frac{1 + F'(0)l}{\rho + g(\gamma-1) + \frac{\gamma(1-\gamma)\sigma^2}{2} - \frac{l\gamma\rho}{1+l}}. \quad (27)$$

When $l = 0$, one can check that $F(0)$ is the equilibrium Price/Dividend ratio for the economy with the specialists as the representative agent. However because in our model the growth of the household sector affects the pricing kernel, this boundary P/D ratio $F(0)$ also depends on the household's labor income l . As in the case where $l = 0$, for the P/D ratio to be well defined we require that parameters satisfy,

$$\rho + g(\gamma-1) + \frac{\gamma(1-\gamma)\sigma^2}{2} - \frac{l\gamma\rho}{1+l} > 0. \quad (28)$$

Furthermore, a straightforward calculation yields that $F'(y^b) = 1$ if $F(y^b) = y^b$. This result also ensures that the mapping from the scaled household's wealth y to the scaled specialist wealth

$w/D = F(y) - y$ is strictly decreasing in the scaled household's wealth y (this monotone relation clearly fails if $F(y^b) > y^b$.) As a result, it is equivalent to model either agent's wealth as our state variable.

A.4 Numerical Method

In our ODE (26) both boundaries are singular, causing difficulties in directly applying the built-in ODE solver *ode15s* in *Matlab*. To overcome this issue, we approximate the upper-end boundary $(y^b, F(y^b) = y^b)$ by $(y^b - \eta, y^b - \eta)$ (where η is sufficiently small), and adopt a “forward-shooting and line-connecting” method for the lower-end boundary. Take a small $\epsilon > 0$ and call \tilde{F} as the attempted solution. For each trial $\phi \equiv \tilde{F}'(\epsilon)$, we set $\tilde{F}'(0) = \phi$, solve $\tilde{F}(0)$ based on (27), and let $\tilde{F}(\epsilon) = \tilde{F}(0) + \phi\epsilon$. Since $(\epsilon, \tilde{F}(\epsilon))$ is away from the singularity, by trying different ϕ 's we apply the standard shooting method to obtain the desired solution F that connects at $(y^b - \eta, y^b - \eta)$. For $y < \epsilon$, we simply approximate the solution by a line connecting $(0, F(0))$ and $(\epsilon, F(\epsilon))$. In other words, we solve F on $[\epsilon, y^b]$ with a smooth pasting condition for $F'(\epsilon) = \frac{F(\epsilon) - F(0)}{\epsilon}$ and a value matching condition for $F(y^b) = y^b$.

We use $\epsilon = 0.1$ and $\eta = 0.001$ which give ODE errors bounded by 3×10^{-5} for $y > \epsilon$. Different ϵ 's and η 's deliver almost identical solutions for $y > 1$. Because we are mainly interested in the solution behavior near y^c (which takes a value of 14 even in the $m = 1$ case) and onwards, our main calibration results are free of the approximation errors caused by the choice of ϵ and η . Finally we find that, in fact, these errors are at the same magnitude as those generated by the capital constraint around y^c (3.5×10^{-5}).

B Verification of optimality

In this section we take the equilibrium Price/Dividend ratio $F(y)$ as given, and verify that the specialist's consumption policy $c = D_t(1 + l - y_t)$ is optimal subject to his budget constraint. Our argument is a variant of the standard one: it uses the strict concavity of $u(\cdot)$ and the specialist's budget constraint to show that the specialist's Euler equation is necessary and sufficient for the optimality of his consumption plan.

Specifically, fixing $t = 0$ and the starting state (y_0, D_0) , define the pricing kernel as

$$\xi_t \equiv e^{-\rho t} c_t^{-\gamma} = e^{-\rho t} D_t^{-\gamma} (1 + l - \rho y_t)^{-\gamma}.$$

Consider another consumption profile \hat{c} which satisfies the budget constraint $E \int_0^\infty \hat{c}_t \xi_t dt \leq \xi_0 D_0 (F_0 - y_0)$ (recall that the specialist's wealth is $D_0 (F_0 - y_0)$; here we require that the specialist's feasible trading

strategies be well-behaved, e.g., his wealth process remains non-negative). Then we have

$$\begin{aligned} E \int_0^\infty e^{-\rho t} u(c_t) dt &\geq E \int_0^\infty e^{-\rho t} u(\widehat{c}_t) dt + E \int_0^\infty e^{-\rho t} u'(c_t)(c_t - \widehat{c}_t) dt \\ &= E \int_0^\infty e^{-\rho t} u(\widehat{c}_t) dt + E \int_0^\infty \xi_t c_t dt - E \int_0^\infty \xi_t \widehat{c}_t dt. \end{aligned}$$

If the specialist's budget equation holds in equality for the equilibrium consumption process c , i.e., if

$$E \int_0^\infty \xi_t c_t dt = \xi_0 D_0 (F_0 - y_0),$$

then the result follows. Somewhat surprisingly, for our model this seemingly obvious claim requires an involved argument because of the singularity at $y^b = \frac{1+l}{\rho}$.

One can easily check that, for $\forall T > 0$, we have

$$\xi_0 D_0 (F_0 - y_0) = \int_0^T c_t \xi_t dt + \int_0^T \sigma(D_t, y_t) dZ_t + \xi_T D_T (F_T - y_T), \quad (29)$$

where $\sigma(D_t, y_t)$ corresponds to the specialist's equilibrium trading strategy (which involves terms such as $(1+l-\rho y)^{-\gamma-1}$ and is NOT uniformly bounded as $y \rightarrow y^b$). Our goal in the following steps is to show that in expectation, the latter two terms vanishes when $T \rightarrow \infty$.

Step 1: Limiting Behavior of y at y^b The critical observation regarding the evolution of y is that when y approaches y^b , it approximately follows a Bessel process with a dimension $\delta = \gamma + 2 > 2$. (Given a δ -dimensional Brownian motion Z , a Bessel process with a dimension δ is the evolution of $\|Z\| = \sqrt{\sum_{i=1}^\delta Z_i^2}$, which is the Euclidean distance between Z and the origin.) According to standard results on Bessel processes, y^b is an entrance-no-exit point, and is not reachable if the starting value $y_0 < y^b$ (if $\delta > 2$). Intuitively, when y is close to y^b , the dominating part of μ_y is proportional to $\frac{1}{y-y^b} < 0$, while the volatility σ_y is bounded—therefore a drift that diverges to negative infinity keeps y away from the singular point y^b . This result implies that our economy never hits y^b .

To show that for y close to y^b , y 's evolution can be approximated by a Bessel Process, one can easily check that when $y \rightarrow y^b$,

$$r \simeq -\frac{(\gamma+1)\sigma^2}{2} \frac{\rho^h \hat{\theta}_b G}{1+l-\rho^h y}, \mu_y \simeq -\frac{(\gamma+1)\sigma^2}{2} \frac{\rho^h \hat{\theta}_b^2 G^2}{1+l-\rho^h y}, \sigma_y = -G\sigma \hat{\theta}_b;$$

and therefore

$$dy = -\frac{(\gamma+1)\sigma^2}{2} \frac{\rho \hat{\theta}_b^2 G^2}{1+l-\rho y} dt - G\sigma \hat{\theta}_b dZ_t.$$

Utilizing the result $F'(y^b) = 1$ established in Section 3.5, we know that when $y \rightarrow y^b$, $\hat{\theta}_b \simeq F - \theta_s y \simeq \frac{1}{1+m} y^b = \frac{1}{1+m} \frac{1+l}{\rho}$, and $G \simeq 1+m$. Let

$$x_t = 1+l-\rho y_t;$$

then it is easy to show that $q = \frac{x}{G\sigma\hat{\theta}_b\rho} = \frac{x}{\sigma(1+l)}$ evolves approximately according to

$$dq = -\frac{1}{G\sigma\hat{\theta}_b}dy = \frac{(\gamma+1)}{2q}dt + dZ_t,$$

which is just a standard Bessel process with a dimension $\delta = \gamma + 2$. Therefore, x is also a scaled version of a Bessel process, and can never reach 0 (or, y cannot reach y^b). In the following analysis, we focus on the limiting behavior of x .

Step 2: Localization Note that in (29), due to the singularity at $x = 0$ (or, $y = y^b$), both the local martingale part $\int_0^T \sigma(D_t, y_t) dZ_t$ and the terminal wealth part $\xi_T D_T (F_T - y_T)$ are not well-behaved. To show our claim, we have to localize our economy, i.e., stop the economy once y is sufficiently close to y^b (or, once D is sufficiently close to 0). Specifically, we define

$$T_n = \inf \left\{ t : x_t = \frac{1}{n} \text{ or } D_t = \frac{1}{n^h} \right\}$$

where h is a positive constant (as we will see, the choice of h , which is around 1, gives some flexibility for γ other than 2). Since y and x have a one-to-one relation ($x = 1 + l - \rho y$), for simplicity we localize x instead.

Clearly this localization technique ensures that the local martingale part $\int_0^{T_n} \sigma(D_t, y_t) dZ_t$ is a martingale (one can check that $\sigma(D_t, y_t)$ is continuous in D_t and y_t , in turn D_t and x_t ; therefore $\sigma(D_t, y_t)$ is locally bounded). As $T_n \rightarrow \infty$ when $n \rightarrow \infty$, for our claim we need to show

$$\lim_{n \rightarrow \infty} E [\xi_{T_n} D_{T_n} (F_{T_n} - y_{T_n})] = 0$$

We substitute from the definition of ξ :

$$E \left[e^{-\rho T_n} D_{T_n}^{1-\gamma} x_{T_n}^{-\gamma} (F(y_{T_n}) - y_{T_n}) \right] \leq E \left[e^{-\rho T_n} n^{h(\gamma-1)} x_{T_n}^{-\gamma} (F(y_{T_n}) - y_{T_n}) \right].$$

Since the analysis will be obvious if $x^{-\gamma} (F(y) - y)$ is uniformly bounded (notice here $x = 1 + l - \rho y$), it is sufficient to consider $x_{T_n} = \frac{1}{n}$. Because $F(y^b) = y^b$ and $F'(y^b) = 1$, by Taylor expansion we know that $F\left(y^b - \frac{1}{n\rho}\right) - \left(y^b - \frac{1}{n\rho}\right)$ can be written as $\psi(n) \frac{1}{n}$ when n is sufficiently large, and $\psi(n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore we have to show that, as $n \rightarrow \infty$,

$$E \left[e^{-\rho T_n} n^{(\gamma-1)(1+h)} \right] \psi(n) \rightarrow 0$$

and a sufficient condition is that,

$$E \left[e^{-\rho T_n} \right] n^{(\gamma-1)(1+h)} \rightarrow K$$

where K is bounded.

We apply existing analytical results in the literature to show our claim. To do so, we have to separate our two state variables. We define

$$T_n^D = \inf \left\{ t : D_t = \frac{1}{n^h} \right\}, T_n^x = \inf \left\{ t : x_t = \frac{1}{n} \right\}.$$

We want to bound $E[e^{-\rho T_n}]$ by the sum of $E[e^{-\rho T_n^D}]$ and $E[e^{-\rho T_n^x}]$; note that they are Laplace transforms of the first-hitting time distribution of a GBM and Bessel processes, respectively. The Laplace transform of T_n is simply

$$E[e^{-\rho T_n}] = \int_0^\infty e^{-\rho T} d\mathbf{F}(T) = \rho \int_0^\infty e^{-\rho T} \mathbf{F}(T) dT,$$

where the bold \mathbf{F} denotes the distribution function of T_n . The similar relation also holds for T_n^D or T_n^x . Denote $\mathbf{F}^D(\cdot)$ (or $\mathbf{F}^x(\cdot)$) as the distribution function for T_n^D (or T_n^x), and notice that

$$\begin{aligned} 1 - \mathbf{F}(T) &= \Pr(T_n > T) = \Pr(T_n^D > T, T_n^x > T) > \Pr(T_n^D > T) \Pr(T_n^x > T) \\ &= 1 - \mathbf{F}^D(T) - \mathbf{F}^x(T) + \mathbf{F}^D(T) \mathbf{F}^x(T), \end{aligned}$$

because $\mathbf{1}_{\{T_n^D > T\}}$ and $\mathbf{1}_{\{T_n^x > T\}}$ are positively correlated (both take the value 1 when Z is high).¹⁵ Therefore $\mathbf{F}(T) < \mathbf{F}^D(T) + \mathbf{F}^x(T)$, or

$$E[e^{-\rho T_n}] n^{(\gamma-1)(1+h)} < E[e^{-\rho T_n^D}] n^{(\gamma-1)(1+h)} + E[e^{-\rho T_n^x}] n^{(\gamma-1)(1+h)}$$

Now we use the standard result of the Laplace transform of the first-hitting time distribution for a GBM process (e.g., Borodin and Salminen (2002), page 622):

$$E[e^{-\rho T_n^D}] = n^{-\frac{h}{\sigma^2}} \left(\sqrt{2\rho\sigma^2 + (g - 0.5\sigma^2)^2} + g - 0.5\sigma^2 \right);$$

therefore when we choose some appropriate h so that

$$\frac{h}{\sigma^2} \left(\sqrt{2\rho\sigma^2 + (g - 0.5\sigma^2)^2} + g - 0.5\sigma^2 \right) > (\gamma - 1)(1 + h),$$

the first term $E[e^{-\rho T_n^D}] n^{(\gamma-1)(1+h)}$ vanishes as $n \rightarrow \infty$. For instance, this condition holds when $h = 0.9$ under our parameterization.

Step 3: Regulated Bessel Process The challenging task is the second term. Notice that our economy (i.e., evolution of x) differs from the evolution of a Bessel process when x is far away from 0; therefore an extra care needs to be taken. We consider a regulated Bessel process which is reflected at some positive constant \bar{x} . Intuitively, by doing so, we are putting an upper bound for $E[e^{-\rho T_n^x}]$, as the reflection makes x_t to hit $\frac{1}{n}$ more likely (therefore, a larger \mathbf{F}^x). Also, for a sufficiently small

¹⁵Technically, using the technique of Malliavian derivatives, we can show that both x_s and D_s have positive diffusions in the martingale representations for all s . Then, the running minimum $\underline{x}_T = \min\{x_t : 0 < t < T\}$ and $\underline{D}_T = \min\{D_t : 0 < t < T\}$ have positive loadings always on the martingale representations (using the technique in *Methods of Mathematical Finance*, Karatzas and Shreve (1998), Page 367). The same technique can be applied to $\mathbf{1}_{\{T_n^x > T\}} = \mathbf{1}_{\{\underline{x}_T > T\}}$ and $\mathbf{1}_{\{T_n^D > T\}} = \mathbf{1}_{\{\underline{D}_T > T\}}$, as an indicator function can be approximated by a sequence of differentiable increasing functions.

$\bar{x} > 0$, when $x \in (0, \bar{x}]$, x can be approximated by a Bessel process with a dimension $\gamma + 2 - \epsilon$. Therefore, \mathbf{F}^x must be bounded by the first-hitting time distribution of a Bessel process with a dimension δ , where δ takes value from $\gamma + 2 - \epsilon$ to $\gamma + 2$, where ϵ is sufficiently small. Finally, note that by considering a Bessel process we are neglecting certain drift for x . However, one can easily check that when x is close to 0, the adjustment term for μ_y is $-\frac{1+l}{\rho}\gamma\sigma^2 < 0$. This implies that we are neglecting a positive drift for x —which potentially makes hitting less likely—thereby yielding an upper-bound estimate.

We have the following Lemma from the Bessel process.

Lemma 1 *Consider a Bessel process x with $\delta > 2$ which is reflected at $\bar{x} > 0$. Let $\nu = \frac{\delta}{2} - 1$. Starting from $x_0 \leq \bar{x}$, we consider the hitting time $T_n^x = \inf \{t : x_t = \frac{1}{n}\}$. Then we have*

$$E [e^{-\rho T_n^x}] \propto n^{-2\nu} \text{ as } n \rightarrow \infty$$

Proof. Due to the standard results in Bessel process and the Laplace transform of the hitting time (e.g., see Borodin and Salminen (1996), Chapter 2), we have

$$E [e^{-\rho T_n^x}] = \frac{\varphi(x_0)}{\varphi(\frac{1}{n})},$$

where

$$\varphi(z) = c_1 z^{-\nu} I_\nu(\sqrt{2\rho}z) + c_2 z^{-\nu} K_\nu(\sqrt{2\rho}z),$$

and $I_\nu(\cdot)$ (and $K_\nu(\cdot)$) is modified Bessel function of the first (and second) kind of order ν . Because R is a reflecting barrier, the boundary condition is

$$\varphi'(\bar{x}) = 0,$$

which pins down the constants c_1 and c_2 (up to a constant multiplication; notice that this does not affect the value of $E [e^{-\rho T_n^x}]$). Therefore the growth rate of $E [e^{-\rho T_n^x}]$ is determined by $n^\nu K_\nu(\sqrt{2\rho}n^{-1})$ as K_ν dominates I_ν near 0. Since $K_\nu(x)$ has a growth rate $x^{-\nu}$ when $x \rightarrow 0$, the result is established.

■

For any y_0 , redefine starting point as $x_0 = \min(1 + l - y_0, \bar{x})$; clearly this leads to an upper-bound estimate for $E [e^{-\rho T_n^x}]$. However, since for all $\delta \in [\gamma + 2 - \epsilon, \gamma + 2]$, the above Lemma tells us that for any $\epsilon \in [0, \epsilon]$, when $n \rightarrow \infty$,

$$n^{(\gamma-1)(1+h)} E [e^{-\rho T_n^x}] \propto n^{(\gamma-1)(1+h)} n^{-2\nu} = n^{(\gamma-1)(1+h)-\gamma+\epsilon} \rightarrow 0$$

uniformly if $\gamma = 2$ and $h = 0.9$ (and for some sufficiently small $\epsilon > 0$). Therefore we obtain our desirable result.

Finally $c_t \xi_t > 0$ implies that $\int_0^\infty c_t \xi_t dt$ converges monotonically, and therefore the specialist's budget equation $\lim_{T \rightarrow \infty} E \int_0^T \xi_t c_t dt = \xi_0 D_0 (F_0 - y_0)$ holds for all stopping times that converge to infinity. Q.E.D.

C Appendix for Section 6

C.1 Borrowing Subsidy

We have the same ODE as in Appendix A. The only difference is that

$$\mu_y = \frac{1}{1 - \theta_s F'} \left(\theta_s + l + (r + \sigma^2 - g) \hat{\theta}_b - \hat{\theta}_b \Delta r - \rho y + \frac{1}{2} \theta_s F'' \sigma_y^2 \right).$$

C.2 Direct Asset Purchase

In this case, the intermediary holds $1 - s$ of the risky asset (where s is a function of (y, D)). In the unconstrained region, $\alpha^h = 1$, and

$$\frac{\alpha^I (w + \alpha^h (1 - \lambda) w^h)}{P} = 1 - s$$

which implies that $\alpha^I = \frac{(1-s)F}{F-\lambda y}$. Therefore the households' holding of the risky asset through intermediaries is

$$\theta_s^I = \frac{(1-s)(1-\lambda)y}{F-\lambda y},$$

and the total holding is $\theta_s = \theta_s^I + s = \frac{(1-s)(1-\lambda)y}{F-\lambda y} + s(y, D)$.

In the constrained region, $\alpha^h = \frac{m(F-y)}{(1-\lambda)y}$ and $\alpha^I = \frac{1}{1+m} \frac{(1-s)F}{F-y}$. So

$$\theta_s^I = \frac{m(F-y)}{(1-\lambda)y} \frac{1}{1+m} \frac{(1-s)F}{F-y} (1-\lambda) \frac{y}{F} = \frac{m}{1+m} (1-s)$$

and the total holding is

$$\theta_s = \frac{m}{1+m} (1-s) + s = \frac{m+s}{1+m}.$$

The same constraint cutoff applies $y^c = \frac{m}{1-\lambda+m} F^c$.

Finally, the formal expressions for the case of capital infusion (i.e., changing m) is isomorphic to the case of $s > 0$.