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7/25/07
9:00 AM

Lotteries in Student Assignment*

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This Version: November 2006
Job Market Paper

Abstract

Public school choice plans across the United States use lotteries to make assignments. Motivated by design issues in the New York City High School match, this paper compares lotteries in the allocation of school seats. The first mechanism, random serial dictatorship, is based on a single lottery: it selects an ordering from a given distribution and assigns the first student her top choice, the second student his top choice among available schools, and so on. The second mechanism, top trading cycles with random priority, is based on lotteries for each school: it selects an ordering from a given distribution and sets that order as the priority for the first school, selects another ordering from the same distribution and sets it as the priority for the second school, and so on. Then the mechanism finds an assignment in the induced market with these priorities using top trading cycles, where cycles form when each student points to the school she desires the most among available schools and each school points to the student in the market who receives the highest priority at that school. This paper shows that a random serial dictatorship is equivalent to top trading cycles with random priority.

*I am grateful to Gary Chamberlain, Drew Fudenberg, Ariel Pakes, and Alvin Roth for guidance and support throughout this project. I am also extremely fortunate to have received detailed comments from Michael Ostrovsky and Tayfun Sönmez. Atila Abdulkadiroğlu, Estelle Cantillon, Fuhito Kojima and Muriel Niederle also provided helpful discussions. For financial support, I am grateful to the National Science Foundation, the Spencer Foundation, the State Farm Corporations, and the Division of Research at Harvard Business School.

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1 Introduction

Current reforms in public education focus on increasing student choice. A central element of school choice is a student assignment plan, a set of rules and procedures used to assign students to the various schools in the choice plan. Recently, the mechanism design literature on the allocation of indivisible goods has influenced the redesign of choice plans in two large school districts. In 2003, New York City's Department of Education (NYC DOE) adopted a new student assignment mechanism to place eighth and ninth graders to public high school (Abdulkadiroğlu, Pathak, and Roth (2005)) and in 2005, the School Committee of Boston Public Schools approved the Superintendent's proposal to change the assignment mechanism used to allocate students to public elementary, middle, and high schools and implemented a new mechanism for the 2005-06 school year (Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006)).[1] To date, more than 300,000 students have been assigned to public school through these new student assignment mechanisms.

The design and implementation of these assignment mechanisms have inspired questions which require refining our theoretical knowledge about matching mechanisms. In school choice problems, one aspect that has been highlighted is the role of indifferences in the orderings of students at schools. In many school districts, students are ordered based on priority classes at schools. In Boston, for instance, students who have siblings that attend the school and who live in a neighborhood walk zone surrounding the school are in the top priority class for the school. Students who only have a sibling at the school are in the next priority class, followed by students who only live in the neighborhood walk zone, and finally the remaining students. Within a priority class, students are ordered based on the outcome of a lottery. Nearly every school district with an open enrollment or controlled choice plan employs lotteries to convert priority classes into a strict ordering of students.[2]

In the design of the New York City High School match, a theoretical issue arose involving the role of lotteries. In formal models of matching markets following Gale and Shapley

[1]See Abdulkadiroğlu, Pathak, and Roth (2006) for research inspired by the New York City experience, and Abdulkadiroğlu and Sönmez (2003), Pathak and Sönmez (2006), and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) for research related to Boston.

[2]A partial list of school districts with choice plans where lotteries are used includes Albany NY, Anchorage AK, Berkeley CA, Boston MA, Brockton MA, Cambridge MA, Champaign IL, Charlotte-Mecklensburg NC, Clarke County GA, Columbus OH, Denver CO, Durham NC, Escambia County FL, Eugene OR, Framingham MA, Irvine CA, Jackson County FL, Lee County FL, Los Banos CA, Malden MA, Miami-Dade FL, New Haven CN, Palo Alto CA, Palm Beach FL, Portland OR, Rochester NY, San Diego CA, San Francisco CA, Seattle WA, St. Lucie FL, Tacoma WA, Tampa-St. Petersburg FL (Hillsborough and Pinellas Counties), Upper Marlboro MD, White Plains NY, Wilmington DE, and Wyandotte WA.

(1962), participants usually submit a rank order list and receive an assignment. When a student is unassigned, it is because she is less preferred by each program she ranked than all of the students who obtained that program. In school choice plans, however, all students have the right to attend school, and unassigned students must be assigned to a school place. In New York City, the number of students who are unassigned after the main round is over 8,000 students, or almost 10% of the entire applicant pool. These unassigned students are asked to submit a new rank order list for the supplementary round. In this round, any high school program with extra capacity may participate, but no program ranks students. This paper analyzes a model which directly corresponds to this assignment problem.

In the supplementary round, without any school priorities, there are many ways to order students via lotteries. Two natural methods are: 1) use one lottery to order each student, or 2) for each school, use a lottery to order each student. During the design phase of the mechanism, officials from the NYC DOE believed that a single lottery is less equitable than using lotteries at each school. In e-mail correspondence, one official stated:

> I believe that the equitable approach is for a child to have a new chance with each [...] program. If we use only one random number, and I had the bad luck to be the last student in line this would be repeated 12 times and I would never get a chance.[3] I do not know how we could explain this to a parent.

After community forums to discuss changes in the new assignment process in December 2003, another DOE official stated:

> Although students might not get their first choices, they were considered separately for each program. There was a rank order established and each student had an equal chance to be selected. One random run creates the same type [as the ordering of Specialized High Schools based on test scores] of line but it is based on the luck of the draw, not a test. If we want to give each child a shot at each program, the only way to accomplish this is to run a new random.
>
> ... I cannot see how the children at the end of the line are not disenfranchised totally if only one run takes place. I believe that one line will not be acceptable to parents. When I answered questions about this at training sessions, (it did come up!) people reacted that the only fair approach was to do multiple runs.

This paper formally investigate this issue by examining two competing mechanisms for student placement. The first mechanism, random serial dictatorship, selects an ordering

[3]In the supplementary round, students are asked to submit a rank order list of up to 12 schools.

from a given distribution and assigns the first student her top choice, the second student his top choice among available schools, and so on. The second mechanism, top trading cycles with random priority, selects an ordering from a given distribution and sets that order as the priority for the first school, selects another ordering from the same distribution and sets it as the priority for the second school, and so on. Then the mechanism finds an assignment in the market with these priorities using top trading cycles.[4] The main result of this paper is that random serial dictatorship is equivalent to top trading cycles with random priority. For any student preference profile, the distribution of student assignments is exactly the same whether there is a single lottery or a lottery at each school, followed by top trading cycles.

The rest of this paper focuses on two other mechanisms, student proposing deferred acceptance mechanism (DA) and probabilistic serial mechanism. DA finds a stable matching for the economy when school priorities are strict. A stable matching is one where there is no student and school pair, where a student prefers the school over her assignment, and the school gives the student a higher priority than a student who is assigned to the school. When there are no school priorities, using a single lottery to order students, and then using this ordering to define school preferences for each school is equivalent to a serial dictatorship for this ordering. Using a lottery for each school to define school preferences is not equivalent to a serial dictatorship, since DA may select a matching which is stable with respect to these school priorities, but is not efficient while a serial dictatorship always selects an efficient matching. When I compare these two lottery methods under DA based on a student's likelihood of receiving her top k choices for all k, there is no relationship between single lotteries and multiple lotteries under DA for all student preference profiles. To investigate the differences between single and multiple lotteries under DA, I use data on stated choices from the supplementary round in 2003-04. With these stated preferences, I show that the efficiency costs are significant. While 60.6% of students receive their top choice under a single lottery, only 47.2% of students receive their top choice under multiple lotteries. Moreover, for the stated preferences, the cumulative likelihood of receiving the top k choices for all k is higher under a single lottery than multiple lotteries.

Both top trading cycles with random priority and deferred acceptance with any lottery to order students are strategy-proof mechanisms, where a student can do no worse than

[4] Top trading cycles is a procedure that has been usually defined for the housing market (Shapley and Scarf (1974)), where each agent is endowed with a house. In school choice problems, when schools have priorities, cycles form when each student points to her favorite school among those available, and each school points to the student in the market who has the highest priority at the school. The next section precisely defines top trading cycles for the school choice problem.

submitting her true ordinal preferences to the mechanism regardless of what other students do. However, when students are endowed with von-Neumann Morgenstern utility functions, neither mechanism is ex ante efficient. Since there is no strategy-proof, anonymous, ex ante efficient mechanism (Zhou (1990)), if a school district is willing to consider a mechanism that is not strategy-proof, the district may be able to improve upon the efficiency of a random serial dictatorship. The probabilistic serial mechanism described in Bogomolnaia and Moulin (2001) finds an ordinally efficient matching, one that is not stochastically dominated by any other matching. Using stated preferences from the supplementary round in NYC in 2003-04, 50% of participants in the supplementary round would receive a distribution of assignments under probabilistic serial which stochastically dominates the distribution from a random serial dictatorship, while only 6% of participants receive a distribution from a random serial dictatorship which stochastically dominates probabilistic serial. For the majority of students, however, the magnitude of the difference between probabilistic serial and random serial dictatorship is negligible.

Related literature

The canonical model of indivisible goods allocation is the housing market model of Shapley and Scarf (1974), where every student in the market occupies a house. Shapley and Scarf describe the top trading cycles procedure (attributed to Gale) to find the core of this market. For this model, Roth and Postlewaite (1977) show that the unique core allocation is equivalent to the competitive equilibrium, and Roth (1982) shows that the core, as a direct mechanism, is strategy-proof. Ma (1994) further demonstrates that the core based mechanism is the only mechanism that is Pareto efficient, individually rational, and strategy-proof.

When all houses are objects to be assigned, with no existing property rights, the model is known as the house allocation problem (Hylland and Zeckhauser (1979)). Svensson (1999) shows that the only mechanism that is strategy-proof, non-bossy, and neutral is a serial dictatorship. There are other characterizations of serial dictatorships as well. The class of lottery mechanisms which are known as random serial dictatorship or random priority is studied by Zhou (1990) and Abdulkadiroğlu and Sönmez (1998). Zhou (1990), proving a conjecture of Gale, shows that there is no anonymous, strategy-proof, and ex ante efficient mechanism for this domain. Bogomolnaia and Moulin (2001) show how the distribution of matchings from a random serial dictatorship can be stochastically dominated by another distribution of matchings, and describe an algorithm to compute an ordinally efficient matching, one that is not stochastically dominated by any other matching, for a stated preference profile.

This paper is most closely related to Abdulkadiroğlu and Sönmez (1998), who establish the equivalence between random serial dictatorship and the core from random endowments in house allocation problems. Since top trading cycles with random priority and the core from random endowments are different mechanisms, their result and the result of this paper are distinct. The main result of this paper implies that the following mechanisms are equivalent: 1) top trading cycles with random priority, 2) random serial dictatorship and 3) core from random endowments. In the core-based mechanism with random endowments, if there are n students, then each student is endowed with only one school and will only exit the market in a cycle involving this school. Under this interpretation, schools do not point to any students and students only directly point to one another. When priorities are drawn randomly for each school, however, the same student can obtain the highest priority at multiple schools, and only one of these schools will be part of the cycle in which this student leaves the market. When this student leaves the market, the schools who had been pointing to this student in the previous step will point to the next highest priority student left in the market. When each student who receives the top priority for a school is different, then it is as if each student is endowed with a particular school as in the core-based mechanism. However, under random priorities, there are many other possible priority orderings to consider. While there are $n!$ potential endowments, there are $(n!)^n$ possible priorities for the schools in the market, which introduces considerable complications.[5]

More generally, this paper is connected to a recent market design literature where experience in the field designing mechanisms poses new theoretical puzzles and motivates work towards their resolution. Milgrom (2006), Roth (2002), and Wilson (2002) contain surveys of parts of the market design literature. Abdulkadiroğlu and Sönmez (2003) formally introduce the school choice problem, and Boston and New York City are the first two school districts to employ mechanisms described in their paper. Both of these districts employ mechanisms which are variants of the student proposing deferred acceptance mechanism which has important antecedents in the literature on labor market clearinghouses described in Roth and Sotomayor (1990).

Two related papers in school choice are in the spirit of this research program. Using historical data from Boston Public Schools, Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006) identify at least two types of players in the old Boston mechanism: those who made the mistake of ranking two overdemanded schools, and those who avoided this mistake or

[5]One direction this type of result has been extended is to a house allocation model with existing tenants. In this model, Sönmez and Ünver (2005) examine lottery mechanisms and establish the equivalence between a core-based mechanism where newcomers are randomly endowed vacant houses, and a version of top trading cycles known as You-Request-My-House-I-Get-Your-Turn (Abdulkadiroğlu and Sönmez (1999)).

belonged to parent groups who met to trade advice on strategic behavior. Motivated by this empirical evidence and the comments of Boston Superintendent Thomas Payzant that a strategy-proof mechanism “levels the playing field,” Pathak and Sönmez (2006) analyze the preference revelation game induced by the Boston mechanism when there are sincere (who report their preferences truthfully) and sophisticated (who best respond) players and compare it to the outcome under Boston’s new mechanism, based on the student proposing deferred acceptance mechanism. Their paper shows that sophisticated players will weakly prefer the Pareto dominant Nash equilibrium of the preference revelation game induced by the Boston mechanism over their outcome from student proposing deferred acceptance.

The design environment in New York City leads to a different set of theoretical challenges. In NYC, since a large fraction of high schools do not actively rank students, and instead employ a lottery to order students, the student proposing deferred acceptance mechanism does not necessarily produce a student optimal stable matching. The mechanism produces a stable matching with respect to the ordering of students induced by the lottery, and this may have an efficiency consequence. Abdulkadiroğlu, Pathak, and Roth (2006) analyze a matching model corresponding to this environment, and establish that a student-proposing deferred acceptance mechanism using a single tie breaking order is not dominated by any other strategy-proof mechanism. This result justifies the use of student proposing deferred acceptance mechanism with a single tie breaking rule if the school district desires a strategy-proof and stable matching. This mechanism is the minimal compromise of efficiency which preserves strategyproofness among stable mechanisms. The topic of indifferences in deferred acceptance is also analyzed by Erdil and Ergin (2005), who identify a polynomial time algorithm to compute the student optimal matching when there are indifferences.

2 Model

2.1 School choice problem

Abdulkadiroğlu and Sönmez (2003) define a **school choice problem** as one where there are a number of students each of whom should be assigned a seat at one of a number of schools. Each student has a strict preference ordering over all schools and each school has a strict priority ranking of all students. Each school has a maximum capacity but there is no shortage of the total number of seats.

Formally, a school choice problem consists of:

1. a set of students $I = \{i_1, ..., i_n\}$,

2. a set of schools $S = \{s_1, ..., s_m\}$,

3. a capacity vector $q = (q_{s_1}, ..., q_{s_m})$,

4. a list of strict student preferences $P_I = (P_{i_1}, ..., P_{i_n})$, and

5. a list of strict school priorities $\pi = (\pi_{s_1}, ..., \pi_{s_m})$.

Here sP_is' means that student i strictly prefers school s to school s'. I will assume that every student prefers a school s to being unassigned. q_s denotes the capacity of school s where $\sum_{s\in S} q_s \geq |I|$, and π_s denotes the strict priority ordering of students at school s. If $\pi_s(i) < \pi_s(j)$ for some $i, j \in I$, then student i receives higher priority than student j at school s. For the rest of this paper, I will hold the q vector fixed, so I do not need to carry along additional notation. I will also focus on a problem where the school district places no restriction on the ordering of students and uses lotteries to determine strict school priorities π, as in the Supplementary Round in New York City.

The outcome of a school choice problem is a **matching** $\mu : I \to S$, a function from the set of students to the set of schools such that no school is assigned to more students than its capacity. Let $\mu(i)$ denote the assignment of student i under matching μ.

A student assignment **mechanism** is a systematic procedure that selects a matching for each school choice problem. A student assignment mechanism is a direct mechanism if it requires students to reveal their preferences over schools and selects a matching based on these submitted preferences and student priorities. Formally, a direct mechanism is a function φ which associates a matching with each problem (P_I, π_S). For $i \in I$, let $\varphi_i(P_I, \pi_S)$ denote student i's match under φ given (P_I, π_S).

A mechanism is **strategy-proof** (dominant-strategy incentive compatible) if for every (P_I, π_S) pair, $\forall i$, $\forall \hat{P}_i$, and $\forall Q_{-i}$, $\varphi_i((P_i, Q_{-i}), \pi_S) R_i \varphi_i((\hat{P}_i, Q_{-i}), \pi_S)$ where R_i is the weak preference ordering consistent with P_i.

A matching is **Pareto efficient** if there is no other matching which assigns each student a weakly better school and at least one student a strictly better school. A mechanism is Pareto efficient if for every school choice problem, it selects a Pareto efficient matching.

Let $\mathcal{M}$ be the set of all matchings. A **stochastic mechanism** maps (P, π) to a probability distribution on $\mathcal{M}$. A stochastic mechanism is strategy-proof if it is dominant-strategy incentive compatible when students submit their ordinal ranking over schools. A stochastic mechanism is Pareto efficient if it places positive probability only on matchings that are Pareto efficient for all (P, π).

Abdulkadiroğlu and Sönmez (2003) proposed two competing mechanisms for school choice problems: the top trading cycles mechanism and the student optimal deferred acceptance mechanism. I will begin by discussing top trading cycles, and then discuss the student optimal deferred acceptance mechanism in section 4.

2.2 Top Trading Cycles Mechanism

Before defining the top trading cycles mechanism, I briefly mention where the mechanism has been either used or thoroughly considered by a school district for use. In May 2006, for the first time ever, top trading cycles was employed to assign over 400 students in the after-market in New York City's High School match. The after-market involved over 5,000 students who appealed their assignment. Each of these students submitted a rank order list of up to three schools they would prefer to attend, and top trading cycles played a role in assigning a subset of these students.[6]

Top trading cycles also played a role in the policy discussion in Boston about student assignment. First, a student task force in charge of making recommendations to the Boston Public Schools committee, in their September 2004 report, strongly recommended that Boston change their assignment mechanism to top trading cycles immediately.[7] Second, during school committee deliberation over the student assignment mechanism, top trading cycles was one of two mechanisms discussed in public hearings. The school committee eventually adopted a mechanism based on deferred acceptance, based on a concern that sibling priority should not be used as a tradable priority for students as in top trading cycles. For more details on the empirical case against Boston Public School's old student assignment mechanism, their decision to change their assignment mechanism, and the school committee's view on top trading cycles, see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006).

In a school choice problem, top trading cycles must be defined to account for the fact that schools have a collection of seats, and are not simply houses with only one occupant. Abdulkadiroğlu and Sönmez (2003) introduce counters in their description of top trading cycles, and define top trading cycles for the school choice problem as follows:

Step 1: Assign a counter for each school which keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools.

[6]The use of top trading cycles in this assignment problem is part of ongoing design work in New York City which will be described in future work.

[7]For the complete report from the student task force, see http://boston.k12.ma.us/assignment/TFreport.pdf. The recommendation of the task force in favor of top trading cycles appears on page 15.

Each student points to her favorite school under her announced preferences. Each school points to the student who has the highest priority for the school. Since the number of students and schools is finite, there is at least one cycle. (A cycle is an ordered list of distinct schools and distinct students $(s_1, i_1, s_2, ..., s_k, i_k)$ where s_1 points to i_1, i_1 points to s_2, ..., s_k points to i_k, i_k points to s_1.) Moreover, each school can be part of at most one cycle. Similarly, each student can be part of at most cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduced to zero, the school is also removed. Counters of the other schools are not changed.

In general, at

Step t: Each remaining student points to her favorite school among the remaining schools and each remaining school points to the student with highest priority among the remaining students. There is at least one cycle. Every student in a cycle is assigned a seat at the school that she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero the school is also removed. Counters of all other schools are not changed.

The algorithm terminates when all students are assigned a seat.

Abdulkadiroğlu and Sönmez (2003) show that this version of top trading cycles is both strategyproof and Pareto efficient. I will refer to this definition of the mechanism as AS-TTC.

For the school choice problems in this paper, I will define another version of top trading cycles which is *outcome equivalent* to the version I have just described. This will simplify the analysis to focus on a market where each school has one seat. When I refer to school specific randomization in this market, it actually refers to school-seat specific randomization in the original market.

For each school $s \in S$, without loss of generality, index seats so that there is a first seat, second seat, and so on.

Step 0: (Initialization) For each school $s_k \in S$, define s_k^i to be the ith seat at school s_k, where $i = 1, ..., q_{s_k}$. Set π_{s_k} to be the priority ordering for each school seat s_k^i. For each student $i \in I$, adjust the announced preference list P_i so that every occurrence of school s_k is replaced by $\{s_k^1, ..., s_k^{q_{s_k}}\}$ in that order. Each school seat can be assigned to at most one student.

Step 1: (Algorithm) Each student points to her favorite school seat under her modified preferences. Each school seat points to the student who has the highest priority. Since the number of students and school seats is finite, there is at least one cycle. Moreover, each school seat at school s will point to the same student under π_s, so each school is part of at most one cycle. Similarly, each student can be part of at most one cycle. Every student in a cycle is assigned the school seat she points to and she and the school seat are both removed.

In general, at

Step t: Each remaining student points to her favorite school seat among the remaining school seats and each remaining school seat points to the student with highest priority among the remaining students. There is at least one cycle. Every student in a cycle is assigned the school seat that she points to and is removed.

The algorithm terminates when all students are assigned a seat.

I will focus on this definition of the mechanism and refer to it as TTC.

Remark 1 *For any economy (P_I, π_S), the outcome of AS-TTC is the same as TTC.*

This remark implies that TTC inherits the properties of AS-TTC, and thus is also strategy-proof and efficient. It is important to note that the priorities considered under multiple lotteries in TTC or in AS-TTC are not equivalent. If there are n students, using counters and AS-TTC means that a school with k seats has $n!$ possible orderings, while under TTC there are $(n!)^k$ orderings. I leave the examination of multiple lotteries in AS-TTC for future work and focus on a problem where each school in the market has unit capacity of 1.[8]

To avoid introducing new notation, I refer to S as the set of school seats, each seat has a corresponding priority order π_s, and P_I as the preferences for students modified to rank school seats. There are n students in the economy and n school seats.

[8]The equivalence of multiple lotteries under AS-TTC and a random serial dictatorship appears to also be true based on extensive simulations.

3 Equivalence for the School Choice Problem

3.1 Definitions

The *choice* $Ch_i(S')$ of an student $i \in I$ from a set of schools $S' \subseteq S$ is the best school among S'. That is,

$$Ch_i(S') = s' \Leftrightarrow s' \in S' \text{ and } s'P_i s \text{ for all } s \in S' \backslash s'.$$

Let $f : \{1, 2, ..., n\} \to I$ be a bijection and $\mathcal{F}$ be the class of all such bijections. Note that $|\mathcal{F}| = n!$. Each bijection is an *ordering* of the students. For any $f \in \mathcal{F}$, student $f(1)$ is first, student $f(2)$ is second, and so on. For any f, $f^{-1}(a) = i$ if and only if $f(i) = a$.

Given any ordering $f \in \mathcal{F}$ of students, define the **serial dictatorship** induced by f, ψ^f as:

$$\begin{aligned}
\psi^f(f(1)) &= Ch_{f(1)}(S), \\
\psi^f(f(2)) &= Ch_{f(2)}(S \backslash \{\psi^f(f(1))\}), \\
&\vdots \\
\psi^f(f(i)) &= Ch_{f(i)}(S \backslash \cup_{j=1}^{i-1} \{\psi^f(f(j))\}), \\
&\vdots \\
\psi^f(f(n)) &= Ch_{f(n)}(S \backslash \cup_{j=1}^{n-1} \{\psi^f(f(j))\}).
\end{aligned}$$

Denote the matching corresponding to the outcome of the serial dictatorship for this ordering of students as m^{ψ^f}.

A **random serial dictatorship** is a stochastic mechanism ψ^{rsd} defined as:

$$\psi^{rsd} = \sum_{f \in \mathcal{F}} \frac{1}{n!} m^{\psi^f}.$$

Each serial dictatorship is selected with equal probability, or equivalently, an ordering is randomly chosen with uniform distribution and the induced serial dictatorship is used.

Recall that $\pi = \{\pi_k\}_{k=1}^n$ is a collection of functions $\pi_k : I \to \{1, 2, ..., n\}$ such that for school k and students $i, j \in I$, $\pi_k(i) < \pi_k(j)$ means that student i is given higher priority than student j at school k. Let Π be the set of all collections of functions. Note that $|\Pi| = (n!)^n$. Denote the matching corresponding to the outcome of top trading cycles in a market with priorities π as m^{φ^π}.

Remark 1. For any f, if for all schools s, π_s is consistent with the same ordering as f (i.e. $\pi_k(f(i)) = i, \forall i, \forall k$), then TTC yields the same matching as a serial dictatorship with ordering f.

Given any priority structure, TTC will produce a Pareto efficient allocation. Consider the following stochastic mechanism **TTC with random priority**, φ^{ttc-rp}, defined as:

$$\varphi^{ttc-rp} = \sum_{\pi \in \Pi} \frac{1}{(n!)^n} m^{\varphi^{\pi}}.$$

This mechanism selects each possible priority structure with equal probability and then determines the outcome of TTC for the induced market.

I now state and prove the main result of the paper:

Theorem 1 *For any set of student preferences P_I, $\varphi^{ttc-rp} = \psi^{rsd}$.*

3.2 Overview

The plan of the proof is as follows: For each ordering of students f, I will construct a set of priorities $\Pi(f)$, such that

1) for any $\pi \in \Pi(f), m^{\psi^f} = m^{\varphi^{\pi}}$,
2) for any f, $|\Pi(f)| = (n!)^{n-1}$, and
3) $\forall f_1, f_2 \in \mathcal{F}$, $f_1 \neq f_2$ implies that $\Pi(f_1) \cap \Pi(f_2) = \emptyset$.

The first condition states that the matching which corresponds to the serial dictatorship for the ordering f, m^{ψ^f}, is the same matching as $m^{\varphi^{\pi}}$, the matching produced by top trading cycles, for any priorities $\pi \in \Pi(f)$. The second condition states that the frequency of the matchings under the two mechanisms are consistent. Since $|\mathcal{F}| = n!$, the condition will imply that each ordering f induces a probability $\frac{1}{n!}$ on matching m^{ψ^f} and since $|\Pi| = (n!)^n$, each f corresponds to a set of priorities $\Pi(f)$ with $(n!)^{n-1}$ elements, which induce a lottery with probability $\frac{(n!)^{n-1}}{(n!)^n} = \frac{1}{n!}$ on matching $m^{\varphi^{\pi}}$, where $m^{\varphi^{\pi}} = m^{\psi^f}$. The third condition states that each f defines a unique set $\Pi(f)$, so that there is no double counting. These three conditions together will demonstrate that both mechanisms induce the same probability distribution over matchings.

Since it is possible that for two distinct orderings, $f_1 \neq f_2$, the matchings produced by the serial dictatorships for these two orderings are the same, the main challenge in the construction is to ensure that the set of priorities corresponding to f_1 and f_2 do not overlap.

Consider an example with 3 students, and student preferences as follows:

P_{i_1}	P_{i_2}	P_{i_3}
s_1	s_2	s_1
s_3	s_1	s_3
s_2	s_3	s_2

If the ordering f_1 of students is $i_1 - i_2 - i_3$, then the matching corresponding to the serial dictatorship ψ^{f_1} is

$$m^{\psi^{f_1}} = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

One approach may be to take this matching, examine what schools are obtained by each student, and define the set of priorities which yield the same matching in a way that respects the ordering f_1. This approach might suggest defining the set $\Pi(f_1)$ as follows:

π_{s_1}	π_{s_2}	π_{s_3}
i_1	(i_1)	$\cdot$
$\cdot$	i_2	$\cdot$
$\cdot$	$\cdot$	$\cdot$

At school s_1, student i_1 receives the highest priority and the ordering of students i_2 and i_3 is arbitrary. At school s_2, the ordering is such that either i_1 and i_2 are given the top priority in that order, or i_2 is given the top priority followed by i_1 and i_3 ordered arbitrarily. Finally, at s_3, the ordering of students is arbitrary. In this example, there are 2 priority orderings at s_1, 3 priority orderings at s_2, and 6 priority orderings at s_3 yielding $2 \cdot 3 \cdot 6 = (3!)^2 = 36$ possible arrangements of priorities. Moreover, it is easy to see that for each $\pi \in \Pi(f_1)$, the matching from top trading cycles with priorities π is the same as the matching from the serial dictatorship ψ^{f_1}.

Now suppose the ordering is $f_2 = i_2 - i_1 - i_3$. The matching corresponding to the serial dictatorship ψ^{f_2} is

$$m^{\psi^{f_2}} = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix} = m^{\psi^{f_1}}.$$

Define $\Pi(f_2)$ in a similar manner as before:

π_{s_1}	π_{s_2}	π_{s_3}
(i_2)	i_2	$\cdot$
i_1	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$

At s_1, either i_1 is given the top priority and the remaining students are ordered arbitrarily, or i_2 is given the top priority immediately followed by i_1, and i_3 is given the last priority. At s_2, i_2 receives the top priority, and the other students are ordered arbitrarily, and at s_3 students are ordered arbitrarily. The set of priorities has the appropriate size: $\Pi(f_2) = 36$ and for any $\pi \in \Pi(f_2)$, $m^{\varphi^\pi} = m^{\psi^{f_2}}$. However, it is not true that $\Pi(f_1) \cap \Pi(f_2) = \emptyset$, since both sets have the same elements (where i_1 is given the top priority at s_1 and i_2 is given the top priority at s_2).[9]

The set of priorities $\Pi(f)$ is constructed from the following steps:

1) $(f \to \Delta)$: For each ordering f, define a *set of priority skeletons* Δ. I will refer to an element δ of Δ as a *priority skeleton.* Each priority skeleton can be further broken down into *components* δ_i where $\delta = (\delta_1, ..., \delta_n)$, and there is a component for each of the n schools.
2) $(\Delta \to \alpha)$: For each priority skeleton δ in Δ, define a *priority assignment* α. A priority assignment associates each component $(\delta_1, ..., \delta_n)$ of the priority skeleton with a school $(s_1, ..., s_n)$.
3) $(\alpha \to \Pi)$: Given the priority skeleton in the previous step and priority assignment for that priority skeleton, define the set of priorities for each school based on the component of the priority skeleton assigned to that school.
4) Construct the entire set of priority structures $\Pi(f)$ by repeating steps 2 and 3 for each priority skeleton in the set of priority skeletons Δ.

Loosely speaking, a priority skeleton encodes the essential information used to define the set of priorities for the schools. When I define a priority assignment for each priority skeleton, I will be re-arranging the components of the priority skeleton to associate each component with a school. Once I have associated each component of the priority skeleton with a particular school, I use this assignment to define the set of priority orderings for the school corresponding to this priority skeleton. For a particular f, there are multiple priority skeletons in Δ, so this procedure is repeated for each element of Δ. Each priority skeleton leads to a priority assignment and a set of priorities for the schools.

The requirement that $m^{\psi^f} = m^{\varphi^\pi}$ follows from how priorities are specified based on the priority assignment for a priority skeleton. The condition that $|\Pi(f)| = (n!)^{n-1}$ follows

[9] In this simple example, the set of priorities that I will define specifies that the set of priorities corresponding to f_2 either places i_2 first at s_1 and i_1 first at s_2 with the rest of the students ordered arbitrarily in the remaining spots, or places i_2 first at s_1, i_1 next at s_1, and i_2 first at i_1, with the rest of the student ordered arbitrarily in the remaining spots.

from the structure of the set of priority skeletons. The condition that $f_1 \neq f_2$ implies $\Pi(f_1) \cap \Pi(f_2) = \emptyset$ follows from assigning priorities so that it is possible can recover a unique f based on executing top trading cycles in a particular way. Once the set $\Pi(f)$ is defined, I will demonstrate that there is no overlap by defining the inverse mapping g, and showing that for any $\pi \in \Pi(f)$, $g(\pi) = f$.

The construction of the set $\Pi(f)$ is involved and description of the steps of the construction involves additional notation. To facilitate understanding of these steps, I present the construction together with an example. The appendix also contains a list of symbols used in the construction.

Example

Consider an economy with 8 students and 8 schools. The preferences of the students are as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}
s_3	s_1	s_2	s_1	s_4	s_3	s_1	s_3
$\cdot$	$\cdot$	$\cdot$	s_5	$\cdot$	s_6	s_5	s_8
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	s_7	$\cdot$

While I work in markets where students have strict preferences, I do not specify what follows the rank ordering when I have a $\cdot$ in the table because it is not important for the example. For instance, student i_1's first choice is s_3 and her second and third choice can be any other school in any order. I will work with an ordering of students which places students in order of their index: i_1 is the first student, i_2 is the second student, and so on.

3.3 Construction

For a given ordering f of students and the corresponding serial dictatorship ψ^f, I partition students into sets as follows:

Step 1.) Starting with student $f(1)$, process each student in order until it is the turn of a student a for whom her assignment $\psi^f(a)$ is worse than a school previously assigned to a student. Terminate the first step right before this student. Next proceed to step 2. If such a student does not exist, then f consists of a single step.

In general,

Step t.) Starting with the next student, process students one at a time until it is the turn of a student a for whom her assignment $\psi^f(a)$ is worse than a school previously assigned to a student in the current step t. Terminate step t before student a. Next proceed to step $t + 1$. If such a student does not exist, then the partition consists of t steps.

This procedure defines sets of students such that there is no conflict of interest within a step: no student in a step desires the school assigned to another student who is part of the same step. Moreover, the first student in any step $t > 1$, prefers a school assigned to a student in step $t - 1$ over the school she receives in step t. Finally, the number of students in each step is equal to the number of schools assigned in the step.

Suppose the ordering f leads to a serial dictatorship which consists of T steps. For each $t \leq T$, let S_t denote the schools assigned at step t and let A_t denote the set of students (agents) assigned in step t. Let n_t be the number of students assigned in step t. Define $N_t = \sum_{i=1}^{t} n_i$ as the number of students who have been assigned up to and including step t, where for notational convenience $N_0 = 0$.

The purpose of breaking down the construction into steps is to simplify the description of the procedure to go from priority skeletons to priority assignments to priorities for the schools. With this partition, the set of priority skeletons Δ can be broken down into a set of priority skeletons $\Delta = (\Delta_1, ..., \Delta_T)$ where Δ_t is the set of priority skeleton for the schools in step t. Likewise, the priority assignment and the priorities for the schools will be specified within a step first, and then the same procedure will be used for each step.

For $t > 1$, let A_t^* be the set of students in step t who prefer a school in S_{t-1} to their assignment under ψ^f . These are students who are **unsatiated** at step $t - 1$. That is,

$$A_t^* = \{a \in A_t : sP_a\psi^f(a) \quad \text{for some school } s \in S_{t-1}\}.$$

The remaining students in A_t are **satiated** in step $t-1$; each prefers the school she receives from the serial dictatorship ψ^f to any school assigned in step $t - 1$. For each $t > 1$, A_t^* is non-empty by construction since each step after the first begins with an unsatiated student.

Example

In our example, the procedure will process students in step 1 until the fourth student i_4. Student i_4 prefers s_1 over her assignment, $s_1 P_{i_4} \psi^f(i_4) = s_5$, so she begins a new step. Students i_5 and i_6 will also belong to step 2. Student i_7 prefers s_5 to $\psi^f(i_7) = s_8$, so she begins a new step. Student i_8 is also part of this step. The sets are defined as

$$\begin{array}{lll} A_1 = \{i_1, i_2, i_3\} & & S_1 = \{s_1, s_2, s_3\} \\ A_2 = \{i_4, i_5, i_6\} & A_2^* = \{i_4, i_6\} & S_2 = \{s_4, s_5, s_6\} \\ A_3 = \{i_7, i_8\} & A_3^* = \{i_7\} & S_3 = \{s_7, s_8\} \end{array}$$

The number of students in step 1, n_1, is 3, the number of students in step 2, n_2, is 3, and in step 3, $n_3 = 2$. The cumulative number of students in step 1, N_1, is 3, the cumulative number of students in step 2, N_2, is 6, and in step 3, $N_3 = 8$.

3.3.1 From f to Δ

I now construct the set of priority skeletons which correspond to f. Before doing so, it will be helpful to define the following notation: $\mathbb{P}(a_1, ..., a_k)$ refers to the set of strings[10] (ordered lists without replacement) of students $\{a_1, ..., a_k\}$ of length l where $l = 0, ..., k$. Examples of elements of $\mathbb{P}(a_1, ..., a_k)$ are $\{a_1, a_k\}$, $\{a_1, a_2, ..., a_k\}$, $\{a_k, a_2, a_{k-1}\}$, as well as the empty set (when $l = 0$).

Denote the students in step t as $a_{N_{t-1}+1}$, $a_{N_{t-1}+2}$, ..., $a_{N_{t-1}+n_t}$, where a_i is the ith student in ordering f. To define the set of priority skeletons for f, I begin by defining the components of the priority skeleton corresponding to step t. These components are from the set Δ_t, a collection of n_t strings defined as:

$$
\begin{aligned}
\Delta_t = \Big[&(a_{N_{t-1}+1}),\\
&(\mathbb{P}(a_{N_{t-1}+1}), a_{N_{t-1}+2}),\\
&(\mathbb{P}(a_{N_{t-1}+1}, a_{N_{t-1}+2}), a_{N_{t-1}+3}),\\
&\vdots\\
&(\mathbb{P}(a_{N_{t-1}+1}, a_{N_{t-1}+2}, ..., a_{N_{t-1}+n_t-2}), a_{N_{t-1}+n_t-1}),\\
&(\mathbb{P}(a_{N_{t-1}+1}, a_{N_{t-1}+2}, ..., a_{N_{t-1}+n_t-1}), a_{N_{t-1}+n_t})\Big].
\end{aligned}
$$

The **set of priority skeletons** Δ is the collection of the set of priority skeletons for each step: $(\Delta_1, ..., \Delta_t, ..., \Delta_T)$. Write $(\delta_{N_{t-1}+1}, \delta_{N_{t-1}+2}, ..., \delta_{N_{t-1}+n_t}) \in \Delta_t$ to be the components of a particular priority skeleton δ corresponding to step t. For step t, there are n_t components of the priority skeleton, where the first component is $\delta_{N_{t-1}+1} = (a_{N_{t-1}+1})$, the second component is either $\delta_{N_{t-1}+2} = (a_{N_{t-1}+1}, a_{N_{t-1}+2})$ or $\delta_{N_{t-1}+2} = (a_{N_{t-1}+2})$, the third component is either $\delta_{N_{t-1}+3} = (a_{N_{t-1}+1}, a_{N_{t-1}+2}, a_{N_{t-1}+3})$, $\delta_{N_{t-1}+3} = (a_{N_{t-1}+2}, a_{N_{t-1}+1}, a_{N_{t-1}+3})$, $\delta_{N_{t-1}+3} = (a_{N_{t-1}+1}, a_{N_{t-1}+3})$, $\delta_{N_{t-1}+3} = (a_{N_{t-1}+2}, a_{N_{t-1}+3})$ or $\delta_{N_{t-1}+3} = (a_{N_{t-1}+3})$, and so on. I will use $\delta_i(k)$ to refer to the kth position of δ_i.

[10]Strings are often defined as an ordered list of symbols, where any symbol can occur more than once. Throughout this paper, whenever I use the term string, I will be referring to an ordered list where no symbol can be used more than once.

Let $A(\delta_i)$ be the set of students in δ_i. Finally, it will be convenient to define $a_\ell(\delta_i)$ as the last student in δ_i, student $f(i)$.

Example

The set of priority skeletons for each of the steps is defined as follows:

$$\Delta_1 = \Big[(i_1),\\ (\mathbb{P}(i_1), i_2),\\ (\mathbb{P}(i_1, i_2), i_3)\Big],$$

$$\Delta_2 = \Big[(i_4),\\ (\mathbb{P}(i_4), i_5),\\ (\mathbb{P}(i_4, i_5), i_6)\Big],$$

$$\Delta_3 = \Big[(i_7),\\ (\mathbb{P}(i_7), i_8)\Big].$$

For the first step, consider the components of the priority skeleton $(\delta_1, \delta_2, \delta_3) \in \Delta_1$, where $\delta_1 = (i_1)$, $\delta_2 = (i_1, i_2)$, and $\delta_3 = (i_2, i_3)$. In this case, $A(\delta_1) = \{i_1\}, A(\delta_2) = \{i_1, i_2\}$, and $A(\delta_3) = \{i_2, i_3\}$. For the second step, consider $(\delta_4, \delta_5, \delta_6) \in \Delta_2$, where $\delta_4 = (i_4)$, $\delta_5 = (i_4, i_5)$, and $\delta_6 = (i_6)$. In this case, $A(\delta_4) = \{i_4\}, A(\delta_5) = \{i_4, i_5\}$, and $A(\delta_6) = \{i_6\}$. For the third step, consider $(\delta_7, \delta_8) \in \Delta_3$, where $\delta_7 = (i_7)$ and $\delta_8 = (i_8)$. Then, $A(\delta_7) = \{i_7\}$ and $A(\delta_8) = \{i_8\}$.

3.3.2 From Δ to α

The priority assignment phase takes place for each priority skeleton in Δ. I will focus on the components of the priority skeleton for step t, $(\delta_{N_{t-1}+1}, \delta_{N_{t-1}+2}, ..., \delta_{N_{t-1}+n_t}) \in \Delta_t$. I will re-arrange the components and associate each school assigned in step t with one of the components. The re-arrangement will have the property that once I define priorities based on the particular skeleton, the last student in step t, $a_{N_{t-1}+n_t}$, will not leave the market under top trading cycles before $a_{N_{t-1}+n_t-1}$, and so on. This will allow for the recovery of the ordering of students f from the execution of top trading cycles in a particular way.

The **priority assignment** function α maps the set of schools S to the set $\{1, ..., n\}$. In step t, I will specify the priority assignment of the schools assigned to students in step t, S_t. For each school, this mapping will be used to construct a set of priorities for the school.

Begin with the components of a particular priority skeleton in step t, $(\delta_{N_{t-1}+1}, ..., \delta_{N_{t-1}+n_t}) \in \Delta_t$. For this priority skeleton, partition the set of students A_t into subsets $A_t^1, A_t^2, ..., A_t^s, ...$ such that

1) $\forall i \neq j\ A_t^i \cap A_t^j = \emptyset$,

2) the students in A_t^i are all adjacent to each other in the ordering f, and

3) $\forall i < j$, the students in A_t^i are ordered in f before the students in A_t^j.

There are two key features of the priority assignment. First, the schools assigned to students in A_t^i are assigned before the schools assigned to students in A_t^j for $i < j$. The other key feature needed to define the priority assignment is that each set A_t^i is split into two subsets. The nature of these subsets differs for the first substep and subsequent substeps. In the first substep, $A_t^{1,+}$ are the set of satiated students and $A_t^1 \backslash A_t^{1,+}$ are the set of unsatiated students. The separate treatment of satiated and unsatiated students in the first substep ensures that there is a way to recover the steps. In particular, the priorities are defined such that no satiated student in step $t > 1$ should leave the market under top trading cycles in a cycle only involving other satiated students. If this happens, then it will be difficult to determine whether the students belong to step $t-1$ or step t, and thus difficult to recover the ordering f. The priority assignment will ensure that if there is a satiated student in the first substep, she must be a part of a cycle involving an unsatiated student in the first substep. For subsequent substeps, $A_t^{i,+}$ are the the students for whom there is no student in A_t^{i-1} who appears earlier in the component of the priority skeleton for which the student in $A_t^{i,+}$ is the last member. Loosely speaking, these are the students who are on a *higher position* in the priority skeleton than the other students in A_t^i. The separate treatment of students in $A_t^{i,+}$ ensures that there is a way to recover the ordering of students within a step. If, for instance, I define priorities for students in $A_t^{i,+}$ such that a student in $A_t^{i,+}$ receives the top priority among students in the step for the school she is eventually assigned, then this student may leave the market under top trading cycles in a substep prior to substep i. If this happens, then it will be difficult to determine the ordering of students within a step.

Substep 1.) Consider the set of students who are in the first position in a component of the priority skeleton and such that there are no students after the student in the priority skeleton:

$$\{a \in A_t : \delta_{N_{t-1}+i}(1) = a \quad \text{and} \quad a = f(N_{t-1}+i)\}.$$

From this set of students, find the largest set of students who immediately follow the first student in A_t in the ordering f. Suppose there are n_t^1 such students,

$a_{N_{t-1}+1}, ..., a_{N_{t-1}+n_t^1}$. Place these students into A_t^1. Place the remaining students in the set above into H_t^2 (students at a higher position in the priority skeleton). Let S_t^1 be the set of schools assigned to the students in A_t^1. Formally,

$$S_t^1 = \{s \in S_t \quad \text{s.t. } \psi^f(a) = s \quad \text{for some } a \in A_t^1\}.$$

Define $A_t^{1,+} = [A_t \backslash A_t^*] \cap A_t^1$. These are the satiated students who are in A_t^1. The unsatiated students are $A_t^1 \backslash A_t^{1,+}$. For the first step, each student in A_t^1 is satiated by construction. Since the first student in each step $t > 1$ is unsatiated and the first student must be a member of A_t^1, $A_t^1 \backslash A_t^{1,+}$ is non-empty.

Remark 2 *If $n_t^1 = n_t$, then the set of students in the first substep A_t^1 is equal to A_t. Otherwise, $A_t^1 \subset A_t$ and there is at least one student in A_t^1 who is in the first position in two components of the priority skeleton (there exists at least one pair (δ_i, δ_j) such that $\delta_i(1) = \delta_j(1) = a$ for some $a \in A_t^1$.)*

In step 1, since there are no unsatiated students, the priority assignment is slightly different than in step $t > 1$. For step 1, order the schools in $S_1^1 = \{s_1^1, s_1^2, ..., s_1^{n_1^1}\}$ from smallest index to largest index. For each student in A_1^1, set $\alpha(s_1^1) = 1, \alpha(s_1^2) = 2$, and so on. The set of priorities that correspond to school s_1^1 will be based on the component of the priority skeleton δ_1, the priorities which correspond to school s_1^2 will be based on the component of the priority skeleton δ_2, and so on.

In step $t > 1$, define priority assignments as follows:

i) Unsatiated students in A_t^1

For each unsatiated student $a \in A_t^1 \backslash A_t^{1,+}$, find the school that is assigned to the next student in f in A_t^1. If such a student does not exist, find the school that is assigned to the first student in f in A_t^1. Place this school into the set $S_t^{1,*}$. Repeat this for each student in $A_t^1 \backslash A_t^{1,+}$. Order the schools in $S_t^{1,*}$ based on their index. Order students in $A_t^1 \backslash A_t^{1,+}$ based on their order in f. For the first school in $s \in S_t^{1,*}$ and first student $a \in A_t^1 \backslash A_t^{1,+}$, set $\alpha(s) = f^{-1}(a)$. For the second school in $s' \in S_t^{1,*}$ and second student $a' \in A_t^1 \backslash A_t^{1,+}$, set $\alpha(s') = f^{-1}(a')$, and so on for every student in $A_t^1 \backslash A_t^{1,+}$.

ii) Satiated students in A_t^1

Next, find the first satiated student in $a \in A_t^{1,+}$ according to f. If the student is not the last student in A_t^1, find the school s that is assigned to the next student in A_t^1, and set $\alpha(s) = f^{-1}(a)$. If a is the last student A_t^1, then find the school s assigned to the first student in A_t^1 and set $\alpha(s) = f^{-1}(a)$. Continue processing the students in $A_t^{1,+}$ in this way.

At this point, there will be a priority assignment corresponding to each school in S_t^1. Since there are n_t^1 schools, n_t^1 out of n_t components of the priority skeleton are processed.

The next step is to update the priority skeleton δ. For each student $a \in A_t^1$, remove the component of the priority skeleton δ_i where $a_\ell(\delta_i) = a$. Next, for the remaining components of the priority skeleton, remove all students in A_t^1. This leads to a new priority skeleton δ^2, where there are n_t^1 less components components, and in step t the components are $(\delta_{N_{t-1}+n_t^1+1}, ..., \delta_{N_{t-1}+n_t})$.

For the remainder of substeps in step t, proceed as follows:

Substep s.) Consider the set of students who are in the first position in a component of the updated priority skeleton δ^s such that there are no students after the student in the component:

$$\{a \in A_t : \delta_i(1) = a \quad \text{and} \quad a = f(i) \text{ for some component } \delta_i \text{ of } \delta^s\}$$

Take this set of students together with students in H_t^s, and find the largest set of students who immediately follow the last student in A_t^{s-1} in the ordering f. Suppose there are n_t^s such students and place them into set A_t^s. Place the remaining students into H_t^{s+1}.

If A_t^s is empty, then the procedure ends.

Otherwise, let S_t^s be the set of schools assigned to students in A_t^s. Define $A_t^{s,+} = A_t^s \cap H_t^s$. The students in $A_t^{s,+}$ are the subset of students in A_t^s who are on a higher position in the priority skeleton than students in $A_t^s \backslash A_t^{s,+}$.

For each $a \in A_t^s \backslash A_t^{s,+}$, find the school that is assigned to the next student in f in A_t^s. Place these schools into the set $S_t^{s,*}$. For the first student a in $A_t^s \backslash A_t^{s,+}$, find the lowest indexed school s in $S_t^{s,*}$, and set $\alpha(s) = f^{-1}(a)$. Continue in this way for the second student in $A_t^s \backslash A_t^{s,+}$ and so on.

For each $a \in A_t^{s,+}$ if a is not the last student in A_t^s, find the school s assigned to the next student in A_t^s and set $\alpha(s) = f^{-1}(a)$. Otherwise, if the student is the last student in A_t^s, find the school s assigned to the first student in A_t^s and set $\alpha(s) = f^{-1}(a)$.

Update the priority skeleton to δ^{s+1} by removing the components of the priority skeleton corresponding to each $a \in A_t^s$. Next, for the remaining components of the priority skeleton, remove all students in A_t^s. This leads to a new priority skeleton δ^{s+1} which has n_t^s fewer components than δ^s.

Since at least one school is given a priority assignment via α at each substep and at least one δ_i will be removed from the priority skeleton δ that the procedure operates on, the procedure will terminate in a finite number of steps. At the completion of this process for step t, there will be a priority assignment for each each $s \in S_t$ to $\{N_{t-1}+1, N_{t-1}+2, ..., N_{t-1}+n_t\}$.

Example

Consider the components of the priority skeleton for the first step: $\delta_1 = (i_1)$, $\delta_2 = (i_1, i_2)$, and $\delta_3 = (i_2, i_3)$. Since $A_1^1 = \{i_1\}$, $n_1^1 = 1$ and $S_1^1 = \{s_3\}$. Since there is only one school in S_1^1, $\alpha(s_3) = 1$. Update the components of priority skeletons for the step by removing i_1 and δ_1, leaving (δ_2', δ_3') where $\delta_2' = (i_2)$ and $\delta_3' = (i_2, i_3)$. Next move to substep 2. In this substep, $A_1^2 = \{i_2\}$, $S_1^2 = \{s_1\}$, $H_1^2 = \emptyset$. In this case, set $\alpha(s_1) = 2$. Update the components of the priority skeleton in the step by removing δ_2' and every student in A_1^2, yielding priority skeleton component $\delta_3'' = (i_3)$. Finally, in step 3, $A_1^3 = \{i_3\}$, and this implies that $\alpha(s_2) = 3$.

Examine the components of the priority skeleton for the second step: $\delta_4 = (i_4)$, $\delta_5 = (i_4, i_5)$, and $\delta_6 = (i_6)$. Begin by identifying that i_4 and i_6 are both in the first position in a component of the priority skeleton. Since i_6 does not immediately follow i_4 in the ordering, $A_2^1 = \{i_4\}$ and $H_2^2 = \{i_6\}$. The school assigned in the first substep of step 2 is $S_2^1 = \{s_5\}$. Since there is only one school in S_2^1, set $\alpha(s_5) = 4$. Update the components of the priority skeleton by removing i_4 and δ_4, leaving (δ_5', δ_6') where $\delta_5' = (i_5)$ and $\delta_6' = (i_6)$. In this step, $A_2^2 = \{i_5, i_6\}$. Moreover, $H_2^2 = \{i_6\}$. In this case, i_6 is on a higher level than i_5. For each student in $A_2^2 \backslash A_2^{2,+} = \{i_5\}$, find the school that is assigned to the next student in f in A_2^2. The next student is i_6 and the school she is assigned is s_6. Set $\alpha(s_6) = 5$. For each student in $A_2^{2,+} = \{i_6\}$, i_6 is the last student in A_2^2, so find the school assigned to the first student in A_2^2. This student is i_5 and the she receives school is s_4. Therefore, $\alpha(s_4) = 6$.

Finally, examine the components of the priority skeleton for step 3: $\delta_7 = (i_7)$ and $\delta_8 = (i_8)$. Notice that $A_3^1 = \{i_7, i_8\}$ and $A_3^{1,+} = \{i_8\}$. First process students in $A_3^1 \backslash A_3^{1,+} = \{i_7\}$. For this student, find the school that is assigned to the next student in A_3^1. The next student is i_8 and the school she receives in s_8, so set $\alpha(s_8) = 7$. Next, process students in $A_3^{1,+}$. Since i_8 is the last student in A_3^1, find the first student in A_3^1. This student is i_7 and the school she receives is s_7. Thus, set $\alpha(s_7) = 8$.

To summarize, the priority assignment for each element of the priority skeleton is defined as:

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ 2 & 3 & 1 & 6 & 4 & 5 & 8 & 7 \end{pmatrix}.$$

3.3.3 From α to Π

For step t, for the particular components of a priority skeleton in the step $(\delta_{N_{t-1}+1}, ..., \delta_{N_{t-1}+n_t}) \in \Delta_t$ and the corresponding priority assignment α for the schools in the step, define a set of priorities for the schools in step t as follows:

1) Let $s = \alpha^{-1}(N_{t-1}+1)$ and define Π_s, the set of priorities for school s, as follows:

$$\Pi_s = \Big[\pi_s \quad \text{s.t. } \pi_s(f(N_{t-1}+1)) < \pi_s(j), \quad \forall j \in \cup_{i=t}^{T} A_i \backslash f(N_{t-1}+1)\Big].$$

Any element of this set must have the property that the first student in the step is ordered before any other student in the step (or subsequent steps) at school s.

The priority for this school is consistent with placing any student who belongs to a step $i < t$ anywhere in the ordering. Thus, there is no restriction for students in $\cup_{i=1}^{t-1} A_i$. In the cycles which form in top trading cycles with this priority structure, students in step $i < t$ will have left the market before step t. Therefore, where these students are ordered at schools in step t and afterwards will not change the matching which results from top trading cycles.

In general,

k) Let $s = \alpha^{-1}(N_{t-1}+k)$, and suppose there are $l = |A(\delta_{N_{t-1}+k})|$ non-empty elements in $\delta_{N_{t-1}+k}$. The set of priority structures for school s is:

$$\Pi_s = \Big[\pi_s \quad \text{s.t.} \quad \begin{array}{l} \pi_s(\delta_{N_{t-1}+k}(i)) < \pi_s(\delta_{N_{t-1}+k}(i+1)), \text{ for } i = 1, .., l-1 \\ \pi_s(a_\ell(\delta_{N_{t-1}+k})) < \pi_s(j) \quad \forall j \in \cup_{i=t}^{T} A_i \backslash A(\delta_{N_{t-1}+k}) \end{array}\Big]$$

This set is consistent with giving students in any previous step (those students in $\cup_{s=1}^{t-1} A_s$) priority anywhere for the school. Students in subsequent steps $\cup_{s=t+1}^{T} A_s$ together with the students in $A_t \backslash A(\delta_{N_{t-1}+k})$ must receive priority after $a_\ell(\delta_{N_{t-1}+k})$ in the priority ordering.

Continue processing each school in S_t in this way.

Remark 3 *For any* $(\delta_{N_{t-1}+1}, ..., \delta_{N_{t-1}+n_t})$*, each student* $a \in A_t$ *in step* t *receives a higher priority than all of the students in subsequent steps for at least one school that is given a priority assignment in the step.*

This remark ensures that no student in a subsequent step will obtain higher priority for a school than all students in the current step. One key part of the procedure is to ensure that no student in a subsequent step will be able to attain a school assigned in step t. The only time a student in a subsequent step receives higher priority for a school than a student in the step is when there is another student in the step who receives higher priority than the student in the subsequent step for this school. The exact assignment of priorities for this procedure depends on the priority skeleton $(\delta_1, ..., \delta_n)$ and the corresponding priority assignment α.

At the conclusion of this process, I will have defined a set of priorities for each of the schools involved in step t for a particular selection from Δ_t. Next, construct the set of all priority structures by selecting each element of Δ_t and repeating the same procedure. This will define the set of priority structures assigned to objects in step t which correspond to f. Finally, to define the entire set $\Pi(f)$, repeat the procedure for each step t.

The procedure to construct the set $\Pi(f)$ must have two properties: 1) there is a way to recover the partition of steps from the execution of top trading cycles and 2) there is a way to determine the ordering of students within a step.

To determine what step students belong to, the procedure defines priority structures such that every satiated student among those in A_t^1 must be assigned in a cycle involving an unsatiated student. Under top trading cycles, the unsatiated student will point to a school in step $t-1$ and this will prevent the satiated students in A_t^1 from leaving the market. Since no student in a subsequent step will receive higher priority than the first student in step t, the first student in step t will prevent any subsequent students from leaving the market.

To determine the ordering of students within a step, I define priorities such that under top trading cycles, the students leave the market in order of f, where the only time multiple students leave the market is if they are adjacent to each other in the ordering f. Top trading cycles with any $\pi \in \Pi(f)$ has the property that for step t, the last student in A_t according to the ordering f will not be part of a cycle and receive a school prior to when the second-to-last student in A_t is part of a cycle and receives a school under top trading cycles. If multiple students leave the market in order of f I can recover the ordering of students based on which school pointed to the student in the cycle in which she leaves the market and if any student obtained priority above a student who has already left in the previous substep. This will be the way to recover the ordering f.

Example

The priorities that correspond to the particular priority skeleton and priority assignment is:

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
i_1	i_2	i_1	(A_1)	(A_1)	(A_1)	(A_1)	(A_1)
i_2	i_3	$\cdot$	i_6	i_4	i_4	(A_2)	(A_2)
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	i_5	i_8	i_7
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	

where the notation (A_1) means any student in A_1 can be ordered anywhere at the school. For instance, at school s_6, any students in A_1 can either be ordered in any order above i_4, in between i_4 and i_5 or after i_5. At school s_7, students in A_1 or A_2 can be ordered anywhere. In the set described, note the relationship between the priority skeleton and the set of priorities. The example shows clearly that the set of priorities is defined by re-arranging the components of the priority skeleton in each step.

The size of priorities which correspond to this priority skeleton is:

$$\underbrace{(6)!(6)!(7)!}_{\text{step 1}} \cdot \underbrace{\frac{8!}{(5)}\frac{8!}{(5)}\frac{8!}{(5)(4)}}_{\text{step 2}} \cdot \underbrace{\frac{8!}{(2)}\frac{8!}{(2)}}_{\text{step 3}}$$

This is the set of priorities defined for a particular priority skeleton in $(\Delta_1, \Delta_2, \Delta_3) = \Delta$. To construct the entire set $\Pi(f)$, the procedure must be repeated for each possible priority skeleton in Δ.

3.4 Properties of $\Pi(f)$

Given the construction of $\Pi(f)$, I can establish two properties of this set.

Claim 1. For any $\pi \in \Pi(f)$, $m^{\varphi^\pi} = m^{\psi^f}$.

Proof. In step 1, each $a \in A_1$ prefers $\psi^f(a)$ to all other schools in $\psi^f(A_1)$. Since ψ^f is a serial dictatorship, each $a \in A_1$ must receive her top choice. Consider π restricted to the schools in $\psi^f(A_1)$. Begin with the students who obtain the top priority for any school in $\psi^f(A_1)$. By construction, a subset of these students who are immediately in order following the first student each will receive the top priority at a school that another student in the subset desires. Under φ^π, each student in this subset points to the school they desire and since there is no conflict of interest among these students, a set of cycles forms involving these students. Once this cycle is removed, the top priority for the schools in $\psi^f(A_1)$ who have not yet left the market will be given to students in A_1 who have not yet been assigned. A subset of these students, following the last student who left the market in a cycle in the previous substep, will each point to the school for which another student among those remaining in

A_1 receives the top priority. Under φ^π, a set of cycles will form involving these students as each points to her top choice. Once the schools assigned in these cycles are removed, iterate these arguments for the remaining schools in $\psi^f(A_1)$. By construction, the top priorities will be given to students remaining in A_1 who have not yet been assigned. When each points to their top choice, a subset of these will form cycles and leave the economy until every student in A_1 has been assigned. Thus, for $a \in A_1$, $\psi^f(a) = \varphi^\pi(a)$.

Since each student only receives school, once the assignments in $\varphi^\pi(a)$ for $a \in A_1$ have been finalized, the location of these students in the priorities for any subsequent schools does not matter. In step 2, iterate the arguments from step 1. Every student in A_2 must prefer her assignment over any school remaining in the market. For the priorities π for the schools in $\psi^f(A_2)$, when a cycle forms, the top priority for the remaining schools in the step are given to students in the step who have not yet been assigned. This fact combined with the fact that there is no conflict of interest among students in the step will ensure that each student receives the same school under ψ^f as φ^π.

Once φ^π is fixed for any student in step 2, the argument can be iterated for students in step 3 and so on. This will establish the claim. $\diamond$

The next step is to count the number of priorities assigned for each f.

Claim 2. $|\Pi(f)| = (n!)^{n-1}$ for all $f \in \mathcal{F}$.

Proof. For any f, consider the set $\Pi(f)$ and its restriction to the schools assigned in the first step, $\Pi_1(f)$. For the components of the priority skeleton in the first step, $(\delta_1, ..., \delta_{n_1}) \in \Delta_1$, the priority assignment for the schools in the step α is a bijection from the set of schools in the first step to the set $\{1, ..., n_1\}$. This bijection may be different for another member of Δ_1. To note this dependence on δ and α, I subscript each school's priority by π_α, leaving the dependence of α on δ implicit. The bijective property of α allows me to ignore the exact school that is involved for counting purposes.

The entire set will take a form mimicking the priority skeleton for the step:

$$\Pi_1(f) = \Big[\Pi_1 \text{ s.t. } \pi_\alpha(f(1)) < \pi_\alpha(i) \quad \text{for any } i \in [\cup_{t=1}^T A_t] \backslash [f(1)], \tag{1}$$

$$\pi_\alpha(f(2)) < \pi_\alpha(i) \quad \text{for any } i \in [\cup_{t=1}^T A_t] \backslash [\cup_{j=1}^2 f(j)], \tag{2}$$

$$\vdots$$

$$\pi_\alpha(f(n_1)) < \pi_\alpha(i) \quad \text{for any } i \in [\cup_{t=1}^T A_t] \backslash [\cup_{j=1}^{n_1} f(j)]\Big], \tag{3}$$

where which priority is assigned to what school depends α, which in turn depends on the $(\delta_1, ..., \delta_{n_1})$.

At step 1, consider the possible priorities of the form corresponding to (1). Restriction (1) admits $(n-1)!$ orderings of students, as it specifies only that $f(1)$ receives the top priority for the school. For the next student, restriction (2) admits $(n-1)!+(n-2)!$ orderings, corresponding to whether $f(2)$ receives the top priority and the remaining $(n-1)$ students are ordered arbitrarily or $f(1)$ receives the top priority, $f(2)$ receives the second priority, and the remaining $(n-2)$ students are ordered arbitrarily, respectively. Consider student k in step 1, where $\pi_\alpha(f(k)) < \pi_\alpha(i)$ for any $i \in [\cup_{t=1}^{T} A_t]\backslash[\cup_{j=1}^{k} f(j)]$. This restriction admits $(n-1)!+(k-1)(n-2)!+(k-1)(k-2)(n-3)!+...+(k-1)!(n-k)! = \sum_{l=1}^{k-1} \frac{(k-1)!}{(k-l)!}(n-l)! = \frac{n!}{n-(k-1)}$ orderings. Thus, in step 1, I find that

$$|\Pi_1(f)| = \left[\frac{n!}{n} \cdot \frac{n!}{n-1} \cdot \dots \cdot \frac{n!}{n-(n_1-1)}\right] = \prod_{i_1=0}^{n_1-1} \frac{n!}{n-i_1}.$$

Follow the same reasoning for steps 2,...,T. For general step t, I obtain:

$$|\Pi_t(f)| = \left[\frac{n!}{n-\sum_{j=1}^{t-1} n_j} \cdot \frac{n!}{n-\sum_{j=1}^{t-1} n_j - 1} \cdot \dots \cdot \frac{n!}{n-\sum_{j=1}^{t-1} n_j - n_t - 1)}\right] = \prod_{i_t=0}^{n_t-1} \frac{n!}{n-\sum_{j=1}^{t-1} n_j - i_t}.$$

Finally, taking each step together, I obtain:

$$|\Pi(f)| = \prod_{i_1=0}^{n_1-1} \frac{n!}{n-i_1} \cdot \prod_{i_2=0}^{n_2-1} \frac{n!}{n-n_1-i_2} \cdot \dots \cdot \prod_{i_t=0}^{n_t-1} \frac{n!}{n-\sum_{j=1}^{t-1} n_j - i_t} = (n!)^{n-1},$$

which completes the proof of the claim. ■

When I assign priorities to schools within step t, I am essentially rearranging components in the priority skeleton Δ_t. This is the key to counting the number of elements in a simple way.

3.5 Defining the Inverse Mapping

I now show how to construct an f for a given Π by constructing the inverse mapping. For some f, and $\pi \in \Pi(f)$, I execute top trading cycles in the following way:

Step 1) Simultaneously remove all cycles of students which form. Let C_1^1 be the set of these cycles and let G_1^1 be the set of schools assigned in the cycles in set C_1^1. I will refer to this as the first substep of step 1.

Next, simultaneously remove all cycles that do not involve a student who desires a school assigned to a student in cycle C_1^1. Place these cycles into C_1^2 and let G_1^2 be the corresponding set of schools. Continue simultaneously removing all cycles that do not involve a student who desires a school assigned to a student in the cycles that have taken place in the step.

Suppose there are k_1 substeps of step 1. Define the set of cycles which form in this step as: $C_1 = \{C_1^1, C_1^2, ..., C_1^{k_1}\}$, where cycles in the set C_1^1 form before cycles in the set C_1^2 form, and so on. Let $G_1 = \{G_1^1, G_1^2, ..., G_1^{k_1}\}$ be the corresponding set of schools assigned in these cycles. When each cycle that forms in the market involves a student who prefers a school assigned to a student in G_1, proceed to the next step.

In general,

Step t) Simultaneously remove all cycles and place them into C_t^1. Let G_t^1 be the set of schools assigned in the cycles in C_t^1. Next, simultaneously remove all cycles that do not involve a student who prefers a school assigned to a student in C_t^1 over what she receives in the cycle. Place these cycles into the set C_t^2 and define the corresponding G_t^2. Stop when all cycles involve at least one student who was pointing to a school that was assigned in step t.

Suppose there are k_t substeps of step t. Order the sets of cycles by the substep in which they form: $C_t = \{C_t^1, C_t^2, ..., C_t^{k_t}\}$ and define the corresponding set of schools $G_t = \{G_t^1, G_t^2, ..., G_t^{k_t}\}$. When each cycle that forms in the market involves a student who prefers a school in G_t to what she receives in the cycle, proceed to the next step.

This procedure stops when no more students remain. For each step t, let A_{C_t} be the set of students involved in the collection of cycles C_t and let $A_{C_t^s}$ be the set of students involved in the set of cycles in set C_t^s.

For each step $t > 1$ and substep $s = 1$, define $A^+_{C_t^1}$ as the subset of students in $A_{C_t^1}$ who are satiated: they prefer their school assignment to any school assigned in G_{t-1}.

$$A^+_{C_t^1} = \{a \in A_{C_t^1} \quad \text{s.t.} \quad \varphi^\pi(a) P_a h \quad \text{for all } h \in G_{t-1}\}.$$

Let $G_t^{1,*}$ be the set of schools who point to students who are unsatiated: those in $A_{C_t^1} \backslash A^+_{C_t^1}$ who are pointed to by a school in a cycle in C_t^1.

For each step $t = 1, ..., T$ and substep $s > 1$, let h^a be the school that points to student a in the cycle where student leaves the market. Define the set of students $A^+_{C_t^s}$ as those who are on a higher position in the priority ordering for the school that points to them in the cycle where they leave the market than the students in $A_{C_t^{s-1}}$. More precisely,

$$A^+_{C_t^s} = \{a \in A_{C_t^s} \text{ s.t.} \quad \nexists a' \in A_{C_t^{s-1}} \text{ where } \pi_{h^a}(a') < \pi_{h^a}(a)\}.$$

Let $G_t^{s,*}$ be the set of schools who point to students in $A_{C_t^s} \backslash A^+_{C_t^s}$ in cycles C_t^s. The role of unsatiated students $A_t^1 \backslash A_t^{1,+}$ in the first substep for steps $t > 1$ is analogous to the role of students in $A_t^s \backslash A_t^{s,+}$.

With these sets in hand, it is possible to define the inverse mapping $g : \Pi \rightarrow \mathcal{F}$. For any π,

1.) Construct the sets $\{C_t^1, C_t^2, ..., C_t^{k_t}\}$ and $\{G_t^1, G_t^2, ..., G_t^{k_t}\}$ and the corresponding sets $\{C_t\}$ and $\{G_t\}$. For $t > 1$ and $s = 1$, construct $A^+_{C_t^1}$ and $G_t^{1,*}$. For each $t = 1, ..., T$ and $s > 1$, construct $A^+_{C_t^s}$ and $G_t^{s,*}$.

2.) For any t, order the students in A_{C_t} before the students in $A_{C_{t+1}}$.

3.) For any s and $t > 1$, order the students in $A_{C_t^s}$ before the students in $A_{C_t^{s+1}}$.

4.) Order the students in $A_{C_t^1}$. There is a different procedure for $t = 1$ and $t > 1$.

   a.) Order the students in $A_{C_1^1}$ based on the index of the school seat which points to them in the cycle in C_1^1. The student pointed to by the lowest indexed school will be first, followed by the student pointed to by the second lowest indexed school, and so on.

   b.) For $A_{C_t^1}$, look first at the unsatiated students in $A_{C_t^1} \backslash A^+_{C_t^1}$. Order them based on the index of the school which points to them in a cycle in C_t^1. Without loss of generality, write $A_{C_t^1} \backslash A^+_{C_t^1} = \{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_\ell\}$ as the ordering of students.

   To complete the ordering of students in $A_t^{1,+}$, begin with $\tilde{a}_2$. Find the school that $\tilde{a}_2$ points to, and the student a' whom this school points to in a cycle in C_t^1. If $a' \in A_{C_t^1} \backslash A^+_{C_t^1}$, then this student has already been processed, and move to $\tilde{a}_3$. Otherwise, place a' immediately before $\tilde{a}_2$ and find which school student a' points to. Let a'' be the student who this school points to. If $a'' \in A_{C_t^1} \backslash A^+_{C_t^1}$, then this student has already been processed, and move to $\tilde{a}_3$. Otherwise, place a'' immediately before a' in the sub-order. Proceed in a similar way until encountering a student in $A_{C_t^1} \backslash A^+_{C_t^1}$, at which point proceed to $\tilde{a}_3$. Repeat this procedure for each of the students in $\tilde{a}_3, ..., \tilde{a}_\ell$.

   Finally, consider student $\tilde{a}_1$ and find the cycle she belongs to. Find the school that $\tilde{a}_1$ points to and the student a' that this school points to. If $a' \in A_{C_t^1} \backslash A^+_{C_t^1}$, then this student is already processed and the procedure stops. If $a' \in A^+_{C_t^1}$, then order this student at the very end of the sub-order (that orders students in $A_{C_t^1}$), and find the student a'' who is pointed to by the school that a' points to in the cycle. If $a'' \in A_{C_t^1} \backslash A^+_{C_t^1}$, then this student is already processed and terminate the procedure. If $a'' \in A^+_{C_t^1}$, order a'' before a' and proceed in a similar way until encountering a student in $A_{C_t^1} \backslash A^+_{C_t^1}$. This student will already be handled, so terminate the procedure.

At the conclusion of this process, there will be a unique ordering of all students in $A_{C_t^1}$ for all t.

5.) For $s > 1$, first order the students in $A_{C_t^s} \backslash A^+_{C_t^s}$ based on the index of the school which points to them in a cycle in C_t^s. Without loss of generality, write $A_{C_t^s} \backslash A^+_{C_t^s} = \{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_\ell\}$ as the ordering of students.

To complete ordering of the remaining students in $A^+_{C_t^s}$ begin with $\tilde{a}_2$. Find the school that $\tilde{a}_2$ points to, and the student a' whom this school points to in a cycle in C_t^s. If $a' \in A_{C_t^s} \backslash A^+_{C_t^s}$, then this student has already processed, and move to $\tilde{a}_3$. Otherwise, place a' immediately before $\tilde{a}_2$ and find which school student a' points to. Let a'' be the student who this school points to. If $a'' \in A_{C_t^s} \backslash A^+_{C_t^s}$, then this student has already been processed, and move to $\tilde{a}_3$. Otherwise, place a'' immediately before a' in the sub-order. Proceed in a similar way until encountering a student in $A_{C_t^s} \backslash A^+_{C_t^s}$, at which point proceed to $\tilde{a}_3$. Repeat this procedure for each of the students in $\tilde{a}_3, ..., \tilde{a}_\ell$.

Finally, consider student $\tilde{a}_1$ and find the cycle she belongs to. Find the school that $\tilde{a}_1$ points to and the student a' that this school points to. If $a' \in A_{C_t^s} \backslash A^+_{C_t^s}$, then this student is already processed and the process ends. If $a' \in A^+_{C_t^s}$, then order this student at the very end of the sub-order (that orders students in $A_{C_t^s}$), and find the student a'' who is pointed to by the school that a' points to in the cycle. If $a'' \in A_{C_t^s} \backslash A^+_{C_t^s}$, then this student is already processed and we terminate the procedure. If $a'' \in A^+_{C_t^s}$, order a'' before a' and proceed in a similar way until encountering a student in $A_{C_t^s} \backslash A^+_{C_t^s}$. This student will already be handled, so terminate the procedure.

This process orders the students in $A_{C_t^s}$ in a unique way for all $s > 1$.

At the conclusion of this procedure, there will be an ordering f for a set of priorities π. Next, I show that this procedure uniquely associates the same f for each $\pi \in \Pi(f)$:

Lemma. For some f, $\pi \in \Pi(f)$, $f = g(\pi)$.

Proof. The proof proceeds by relating the steps involved in constructing $\Pi(f)$ to the inverse mapping. From Claim 1, $m^{\varphi^\pi} = m^{\psi^f}$.

Claim. For all t and s, $A_{C_t^s} = A_t^s$.

Proof. Under π, no student in $\cup_{s>t} A_{C_s}$ leaves the market before any student in A_{C_t}. Under π, the first student in A_t who is also the first student in A_t^1 receives the highest priority for a school assigned in a cycle in C_t^1 among all students in A_t^1. When $t > 1$, the first student A_t^1 must be unsatiated. If there are multiple students who receive the top priority among the schools assigned to students in A_t^1, then the construction ensures that each cycle that forms among these students must involve at least one unsatiated student. Therefore, these cycles can be identified when the unsatiated student points to a school that had been assigned to a student in the previous step. Moreover, every satiated student in C_t^1 will not be assigned until the first student in the step has been assigned and this student is unsatiated. Thus, the students in $A_{C_t^1}$ cannot be part of A_{t-1} and must be part of A_t^1. Next, since no student in $\cup_{r>k} A_{C_t^r}$ leaves the market before any student in $A_{C_t^k}$ and the students in $A_{C_t^s}$ for $s > 1$ each prefer the school they receive under C_t^s to the schools assigned in some cycle in $C_t^1, ..., C_t^{s-1}$, these students in $A_{C_t^s}$ must be part of A_t^s. $\diamond$

This claim implies that $A_{C_t} = A_{C_t^1} \cup A_{C_t^2} \cup ... \cup A_{C_t^{k_t}} = \cup_{r=1}^{k_t} A_t^r = A_t$.

Claim. For all t and s, $G_t^s = S_t^s$.

Proof. This follows from the fact that $\varphi^\pi = \psi^f$, $A_{C_t^s} = A_t^s$, and that there is no conflict of interest among students within a step. $\diamond$

This claim implies that $G_t = G_t^1 \cup G_t^2 \cup ... \cup G_t^{k_t} = \cup_{s=1}^{k_t} S_t^s = S_t$. Given $A_{C_t^s} = A_t^s$ and $G_t^s = S_t^s$ it is straightforward to see that the satiated students in the first substep are the same for $t > 1$: $A_{C_t^1}^+ = A_t^{1,+}$. Moreover, the set of students who receive higher priority at a school than a student assigned in an cycle in an earlier substep is equal to the set of students who are at a higher level in the substep: $A_{C_t^s}^+ = A_t^{s,+}$. This also implies that $S_t^{1,*} = G_t^{1,*}$ and $S_t^{s,*} = G_t^{s,*}$.

This leaves us to establish the last claim:

Claim. $g(\pi) = f$

Proof. The proof follows by induction. Begin by examining step 1 and all of its substeps. Suppose $a \in A_1 = A_{C_1}$ is among the students who leave the market as part of a cycle in C_1^1

and that $f(i) = a$ for some i. By construction of π, the school which points to a must be the ith smallest indexed school in S_1^1. Since the mapping g places students in $A_{C_1^1}$ ahead of students in $A_{C_1^2}$ and the order within $A_{C_1^1}$ is based on the index of the school which points to the student, student a should be ordered ith by $g(\pi)$. Continue this argument for each student in $A_{C_1^1}$ to establish that $g(\pi)(i) = f(i)$ for all $i \in A_{C_1^1}$.

Next consider the students in $A_{C_1^2}$. Start with the ordering of students who are not on a higher level. Pick $\tilde{a}_i \in A_1^2 \backslash A_1^{2,+}$ and suppose that f orders $\tilde{a}_i$ ith among students in $A_1^2 \backslash A_1^{2,+}$. Since $S_1^{2,*} = G_1^{2,*}$, the school in $G_1^{2,*}$ who points to $\tilde{a}_i$ must be the ith smallest indexed school in $S_1^{2,*}$. Since g orders students in $G_1^{2,*}$ based on the index of the school which points to them, if a is ith among $A_1^2 \backslash A_1^{2,+}$ according to f, it must be ith according to g. Proceed to order each student in $A_1^2 \backslash A_1^{2,+}$ among themselves in this way. Next consider the students in $A_1^{2,+}$. There are two cases to deal with: 1) $i > 1$ and 2) $i = 1$. In the first case, consider $\tilde{a}_{i-1} \in A_t^2 \backslash A_t^{2,+}$ where $\tilde{a}_{i-1}$ is ordered $(i-1)$th among students in $A_t^2 \backslash A_1^{2,+}$. Find the student a ordered between $\tilde{a}_{i-1}$ and $\tilde{a}_i$ immediately before $\tilde{a}_i$. By construction of π, $\tilde{a}_i$ wants the school that points to a and $a \in A_1^{2,+} = A_{C_1^2}^+$, so g will order her right before $\tilde{a}_i$. Next continue with the student a' who is right before a in f. By construction of π, a wants the school that points to a' and $a' \in A_1^{2,+} = A_{C_1^2}^+$, so g will order her right before a. Continue for each such student between $\tilde{a}_{i-1}$ and $\tilde{a}_i$ to show that these students will be ordered the same under f and $g(\pi)$. In the second case, $\tilde{a}_i$ is the first student. Suppose $\tilde{a}_\ell$ is the last student in $A_1^2 \backslash A_1^{2,+}$. I will demonstrate that the students after $\tilde{a}_\ell$ in f will be ordered the same way under g. Let a be the last student in f in A_1^2. By construction of π, $\tilde{a}_1$ points to the school which points to a, and $a \in A_1^{2,+} = A_{C_1^2}^+$ implies that under g, this student is also the last student in A_1^2. Find the student a' immediately before a and repeat the same argument. This will demonstrate that the students in $A_{C_1^2} = A_1^2$ are ordered in the same way under f and $g(\pi)$. Proceed in the same way for each C_t^s for $s > 2$ to show that each $a \in A_1^s$ is ordered the same way under f and g. This will establish our base case: the students in $A_1 = A_{C_1}$ are ordered the same way under f and $g(\pi)$.

Next, suppose that for each step $r \in 2, ..., t-1$, for any student $a \in A_r$, I have shown that f and $g(\pi)$ have same the order for a. I will show that each $a \in A_t$ will have the same ordering under f and $g(\pi)$. The main issue is the distinction between satiated and unsatiated students in cycles C_t^1.

The next part follows in a similar way as for students in step 1. Consider the students in $A_{C_t^1}$. I will start with the ordering of students who are unsatiated. Pick $\tilde{a}_i \in A_t^1 \backslash A_t^{1,+}$ and suppose that f orders $\tilde{a}_i$ ith among students in $A_t^1 \backslash A_t^{1,+}$. Since $S_t^{1,*} = G_t^{1,*}$, the school in $G_t^{1,*}$ who points to $\tilde{a}_i$ must be the ith smallest indexed school in $S_t^{1,*}$. Since g orders students in $G_t^{1,*}$ based on the index of the school which points to them, if a is ith among $A_t^1 \backslash A_t^{1,+}$ according to f, it must be ith according to g. Proceed to order each student in $A_t^1 \backslash A_t^{1,+}$ in this way. Next consider the students in $A_t^{1,+}$. There are two cases to deal with: 1) $i > 1$ and 2) $i = 1$. In the first case, consider $\tilde{a}_{i-1} \in A_t^1 \backslash A_t^{1,+}$ where $\tilde{a}_{i-1}$ is ordered $i-1$th among students in $A_t^1 \backslash A_t^{1,+}$. Find the student a ordered between $\tilde{a}_{i-1}$ and $\tilde{a}_i$ immediately before $\tilde{a}_i$. By construction of π, $\tilde{a}_i$ wants the school that points to a and $a \in A_t^1 \backslash A_t^{1,+}$, so g will order her right before $\tilde{a}_i$. Next continue with the student a' who is right before a in f. By construction of π, a wants the school that points to a' and $a' \in A_t^1 \backslash A_t^{1,+}$, so g will order her right before a. Continue for each such student between $\tilde{a}_{i-1}$ and $\tilde{a}_i$ to show that these students will be ordered the same under f and $g(\pi)$. In the second case, $\tilde{a}_i$ is the first student. Suppose $\tilde{a}_\ell$ is the last student in $A_t^1 \backslash A_t^{1,+}$. I will demonstrate that the students after $\tilde{a}_\ell$ in f will be ordered the same way under g. Let a be the last student in f in A_t^1. By construction of π, $\tilde{a}_1$ points to the school which points to a, and $a \in A_t^1 \backslash A_t^{1,+}$ implies that under g, this student is also the last student in A_t^1. Find the student a' immediately before a and repeat the same argument. This will demonstrate that the students in $A_{C_t^1} = A_t^1$ are ordered in the same way under f and $g(\pi)$. Proceed in the same way for each C_t^s for $s > 2$ to show that each $a \in A_t^s$ is ordered the same way under f and g. This will cover all students in A_t and show that they are ordered the same way under f and $g(\pi)$. $\diamond$

For any $f_1 \neq f_2$, for all $\pi \in \Pi(f_1)$, $g(\pi) = f_1$ and for all $\pi \in \Pi(f_2)$, $g(\pi) = f_2$. There is no $\pi \in \Pi(f_1) \cup \Pi(f_2)$ as this would imply that $f_1 = g(\pi) = f_2$, which is only true for $f_1 = f_2$. Since Claim 1 showed that $m^{\varphi^\pi} = m^{\psi^f}$ for all $\pi \in \Pi(f)$, Claim 2 showed that $|\Pi(f)| = (n!)^{n-1}$, and I have just shown that $\forall f_1 \neq f_2$, $\Pi(f_1) \cap \Pi(f_2) = \emptyset$, I have shown the equivalence of random serial dictatorship and top trading cycles with random priority.

4 Lotteries with Deferred Acceptance

Another interpretation of the quote from policymakers at the NYC DOE is to conduct multiple school specific lotteries for each school, use these lotteries to set school preferences, and compute a matching using the student proposing deferred acceptance mechanism.

Before discussing this proposal, I first define deferred acceptance for any strict ordering of students at schools:

Step 1) Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following only their priority order. Any remaining proposers are rejected.

In general, at

Step k) Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

The algorithm terminates when no student proposal is rejected and each student is assigned her final tentative assignment.

For any ordering of students at schools, this mechanism is strategy-proof (Dubins and Freedman (1981), Roth (1982)). I have already mentioned that when there is a single lottery and this ordering is set as the preferences for each school, this is equivalent to a serial dictatorship for that ordering. Let DA-STB (deferred acceptance algorithm with single tie breaking) refer to the stochastic mechanism is induced by all possible orderings of students.

Remark 4 *DA-STB* $= \psi^{rsd}$

Let DA-MTB (deferred acceptance algorithm with multiple tie breaking) refer to the mechanism where each school has an independent lottery which is used to set priorities, and then student proposing deferred acceptance used to compute a matching.

It is straightforward to see that for all preference profiles, DA-MTB is not equivalent to DA-STB. DA-MTB will lead to a stable matching with respect to the artificial priorities at schools. Since there may be a tension between stability and efficiency in general, there may be an efficiency consequence.

When two mechanisms are not equivalent, one way to compare mechanisms is by stochastic dominance.[11] Let p_i^k be the probability that student i receives her kth choice. An allocation is a vector of probabilities $p_i = (p_i^1, ..., p_i^n)$ for each item on the rank order list P_i such that $\sum_{k=1}^n p_i^k = 1$. I say that an allocation p_i **stochastically dominates** allocation p_i' for student i if for all $m = 1, ..., n$,

$$\sum_{k=1}^m p_i^k \geq \sum_{k=1}^m p_i'^k.$$

An allocation $p = (p_i)_{i=1}^n$ stochastically dominates allocation $p' = (p_i')_{i=1}^n$ if p_i stochastically dominates p_i' for all i.

Proposition 1 *There is no stochastic dominance relationship between DA-STB and DA-MTB.*

[11] This notion is used by Bogomolnaia and Moulin (2001), for example.

Proof. It suffices to find a preference profile where there is not stochastic dominance for all students. Consider a market with three students i_1, i_2, i_3 and three schools s_1, s_2, s_3 each with one seat. Suppose student preferences are:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \\ i_2 &: s_3 \succ s_1 \succ s_2 \\ i_3 &: s_1 \succ s_3 \succ s_2. \end{aligned}$$

DA-STB induces the following distribution over matchings:

$$\frac{1}{3} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix} + \frac{1}{6} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix},$$

while DA-MTB induces the following distribution over matchings:

$$\frac{1}{4} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix} + \frac{1}{6} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix} + \frac{1}{12} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}.$$

The probability distribution on first choice, second choice, and third choice for student i_3 under DA-STB is $\begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$, while under DA-MTB it is: $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. Since student i_3 is more likely to receive either her first or second choice under DA-MTB than DA-STB, DA-STB does not stochastically dominate DA-MTB. Comparisons of the distribution of matchings for student i_1 and i_2 show that DA-MTB does not stochastically dominate DA-STB. $\diamond$

This observation motivates an empirical comparison of DA-STB to DA-MTB using data from the 2003-04 supplementary round in New York City. Table 1 presents a comparison of the distribution of choices for the 8,255 students who participated in the Supplementary Round. In this year, there were 8,255 students who participated, and they submitted a rank order list of up to 12 schools. Approximately 38% of students who participated this round submitted 12 choices. Between 4-8% of students submitted k choices for $k = 1, ..., 11$. A total of 108 different programs were available for students to rank.

The first three columns present the distribution of outcomes from a random serial dictatorship, which is equivalent to DA-STB, for a different number of iterations. Computational constraints limit the feasibility of calculating all 8,255! orderings, so the first column corresponds to 250,000 different draws from the lottery ordering, the second column corresponds to 500,000 and the third column is 1,000,000 draws. The three columns show that there is no significant difference in the distribution of overall rankings when the number of draws is increased beyond 250,000, so this suggests I have a good approximation of the overall distribution of rankings.

The table shows that nearly 5,000 students or 60.6% of students receive their top choice under a random serial dictatorship. In comparison, the fourth column shows the distribution of rankings from 1,000 different simulations of multiple lotteries. The first row of the table shows that the expected number of students who receive their top choice is 3894.30 or 47.2% of students. The expected number of students who receive their second choice under a random serial dictatorship is approximately 1,489 (18.0%) which is slightly fewer than under multiple lotteries followed by student proposing deferred acceptance, where the expected number is approximately 1,887 (22.9%). The total likelihood that a student receives their top two choices under a random serial dictatorship is 78.6%, while under multiple lottery deferred acceptance it is 70.0%. This relationship holds for a cumulative comparison of each choice, and shows that for the preference profile in 2003-04 in the supplementary round, the aggregate distribution of rankings from a random serial dictatorship does stochastically dominate the aggregate distribution of rankings from multiple lottery deferred acceptance.

However, the distribution of rankings under a random serial dictatorship does not stochastically dominate the distribution under DA-MTB for all students. For 1,349 (16.3%) of students, the distribution of RSD stochastically dominates the distribution from DA-MTB. On average, RSD gives students their top choice 13.4% more frequently than DA-MTB, while RSD gives students one of their top two choices 8.6% more frequently than DA-MTB.

5 Ordinal efficiency

If a school district is willing to consider a mechanism that is not strategy-proof, they may be able to improve upon efficiency. Bogomolnaia and Moulin (2001) introduced another notion of efficiency, **ordinal efficiency**, which is stronger than ex-post efficiency and weaker than ex-ante efficiency. An ordinally efficient matching is one where the probability distribution over matchings is not stochastically dominated by any other matching.

Bogomolnaia and Moulin (2001) introduce an algorithm, **probabilistic serial**, which computes an ordinally efficient matching. The algorithm is based on a "simultaneous-eating" procedure, where for a fixed step size, each student begins by consuming the object that they desire the most. Once enough students have consumed the object so that the consumption shares add to 1, each student then starts consuming the object they prefer next. Once the sum of consumption shares equals 1, students move to the next object that they prefer that has not been fully consumed. The induced shares for each object represent the probability distribution over allocations and this will be ordinally efficient. The probabilistic serial mechanism is not strategy-proof.

In this section, I compare the performance of probabilistic serial to random serial dictatorship in the supplementary round of the New York City high school match in 2003-04. The last column of Table 1 shows the expected distribution of rankings from probabilistic serial.[12] The column shows that probabilistic serial assigns a higher expected number of students to their top choice than a random serial dictatorship. Almost 16 more students on average will receive their top choice under probabilistic serial than a random serial dictatorship. Probabilistic serial also assigns a higher number of students to their 2nd, 3rd, ..., 12th choice, and leaves fewer students unassigned.

Table 2 compares the performance of probabilistic serial to a random serial dictatorship by comparing the allocation of students. The distribution of allocations from probabilistic serial stochastically dominates a random serial dictatorship for 4,126 out of 8,255 students, or 50% of all participants. The distribution of allocations from a random serial dictatorship stochastically dominates the distribution of allocations from probabilistic serial for only 495 students, or 6% of participants. For the remaining, 44% of participants there is no stochastic dominance relationship.

The second and third row of Table 2 present another measure of the differences in allocation. The second row that under probabilistic serial, 67.6% of students have a higher likelihood of obtaining their first choice, while the 28.6% of students have a higher likelihood of obtaining their first choice under a random serial dictatorship. For the remaining 3.8% of students, there is no difference in the likelihood of obtaining a first choice. The last row of the table presents the comparison for likelihood of obtaining both first and second choice. In this case, 58.7% of students have a higher likelihood of obtaining one of their top two choices under probabilistic serial, while only 12.7% of students have a higher likelihood under a random serial dictatorship. These last two rows shows that when attention is restricted to either the top choice, or the top two choices, probabilistic serial may benefit an even larger share of the population.

Finally, Figure 1 presents the difference in likelihood of obtaining first choice under probabilistic serial and a random serial dictatorship (light gray) and the difference of obtaining the first and second choice (dark gray). The figure shows that while there is stochastic dominance for one half of the entire population, the size of the probability difference is small for the majority of participants. Under probabilistic serial, the mean improvement in likelihood of obtaining the first choice is 0.12%, the median is 0.03%, and the standard deviation is 0.24%. The figure shows that this distribution is highly skewed, with a small fraction of

[12]This was calculating via a simultaneous eating algorithm where the step size is 10^{-7}. There was only a slight difference between step size 10^{-6} and step size 10^{-7}

students near the maximum improving the likelihood of obtaining their top choice over 2%.

Similarly, when looking at the difference in likelihood of obtaining the first or second choice, the mean difference is 0.40%, the median is 0.20%, and the standard deviation is 0.62%. This distribution is also highly skewed, with the 90th percentile improving their odds by 1.18% and the maximal student improving their odds by 16.6%.

Implications for design

The fact that probabilistic serial produces a lottery over allocations which stochastically dominates the lottery produced by random serial dictatorship for half of participants might suggest that the NYC DOE should reconsider using a random serial dictatorship in the supplementary round. There are at least two tradeoffs. Since 6% of students obtain an allocation from a random serial dictatorship that stochastically dominates the probabilistic serial, the NYC DOE would need to evaluate the welfare of these students in comparison to the 50% of participants who benefit from probabilistic serial.

Another tradeoff involves the incentives of probabilistic serial, which is not strategy-proof. From the experience implementing assignment mechanisms in school choice in the field so far, it appears that policymakers value the ability to give clear advice to participants on how to behave. With a mechanism that is not strategyproof, the school district will have a much harder time describing how students should submit their rank ordering.

For instance, in New York, the policy pronouncement by school administrators during the change suggest that being able to emphasize the incentive features of the matching process was extremely important for both policymakers and the public. For instance, NYC School Chancellor Joel Klein stated in the *New York Times* (10/24/03) that the "changes are intended to reduce the strategizing parents have been doing to navigate a system that has a shortage of good high schools." Furthermore, Peter Kerr, another NYCDOE official, wrote in the *New York Times* (11/3/03): "The new process is a vast improvement... For example, for the first time, students will be able to list preferences as true preferences, limiting the need to game the system. This means that students will be able to rank schools without the risk that naming a competitive school as their first choice will adversely affect their ability to get into the school they rank lower." Perhaps more importantly, the NYC DOE can give straightforward advice to families for the main round. In every year since 2003-04, the High School directory makes a point to advise families to express their preferences truthfully. For instance on page (ii) of the 2003-04 directory, for the main round, the advice given to parents is "You must now rank schools very carefully, to reflect your true preferences," while on page (5), the directory advises students to "rank your twelve (12) selections in order of your true preferences."[13]

6 Conclusion

The paper has studied the role of lotteries in resource allocation motivated by a design issue in New York City's High School Match. The problem corresponds to the supplementary round of the NYC match, when schools do not express preferences over students and all students are in the same priority class. For this problem, the intuition of the NYC DOE was that multiple lotteries are more equitable than a single lottery. I show, however, using a single lottery to order students followed by a serial dictatorship for that order is equivalent to using multiple lotteries to construct priorities for each school, and using top trading cycles to find a matching.

The analysis here leaves open many questions. The model assumes that every student is in the same indifference class at each school. The next direction to pursue involves studying the role of lotteries in a more general model with multiple indifference classes. In a problem where there are some existing students who wish to transfer their assignments, and there are some newcomers, simulation evidence suggests that using a single lottery and then the You-Request-My-House-I-Get-Your-Turn (Abdulkadiroğlu and Sönmez (1999)) is equivalent to a version of top trading cycles with school specific lotteries where each school with an existing student gives the existing student the highest priority. I am pursuing this result in the context of designing New York City's appeals process, where existing students are those who wish to retain the rights to their current school.

More generally, in Boston, there are four indifference classes: sibling-walk, sibling, walk, and no priority. For the simplest version of this model, a single lottery is not equivalent to school specific lotteries.[14] Future work I am pursuing will try to identify whether there is a relationship between lotteries for this more general domain.

Other questions motivated by this paper involve different version of top trading cycles.

[13] Policymakers in Boston made similar statements to the public. Superintendent Thomas Payzant in a memo to the School Committee on May 25, 2005 wrote that "A strategy-proof mechanism adds 'transparency' and clarity to the assignment process, by allowing for clear and straightforward advice to parents regarding how to rank schools. More statements about the policy importance of strategy-proofness are presented in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006).

[14] Consider three students and three schools. Suppose $i_1 : s_1 \succ s_2 \succ s_3$, $i_2 : s_3 \succ s_1 \succ s_2$, and $i_3 : s_1 \succ s_3 \succ s_2$, and that students i_1 and i_3 are the in first priority class at s_1 and at school s_2, student i_2 is the only student in the highest priority class. It is easy to demonstrate that a single lottery will yield a different distribution on matchings than school specific lotteries.

It is possible to draw a distinct ordering for each school, rather than school seat, and employ the version of top trading cycles with counters described in Abdulkadiroğlu and Sönmez (2003). Kesten (2005) has suggested another version of top trading cycles which attempts to minimize situations where a matching is not stable. It would be interesting to know if there is any relationship between multiple lotteries in these mechanisms and random serial dictatorship. Finally, for the student proposing deferred acceptance mechanism, there exist characterizations of priority structures for which there is no conflict between stability and efficiency (Ergin (2002), Kesten (2006)). Another mechanism worth considering is a lottery mechanism which randomizes over priority structures in this class and then uses student proposing deferred acceptance. This mechanism will produce an ex post efficient matching for every preference profile, and may lead to a distribution over allocations which is different than a random serial dictatorship.

The design components of this paper leave open a number of issues that are worth examining as we learn more about the institutional constraints and evolution of participant behavior in New York City. In the supplementary round of the match, using stated preferences, one half of participants would receive a distribution over allocations from probabilistic serial which stochastically dominates the distribution they receive from a random serial dictatorship. Despite this difference, the selection of a random serial dictatorship for that round can be justified on incentive grounds. This motivates a need to understand how easy we expect probabilistic serial to be to manipulate theoretically and in practical applications. One step in this direction is a recent paper by Kojima and Manea (2006).

As the matching system in New York City and other cities evolves, we will be able to learn more about institutional constraints which shape the designs, develop better designs, and improve our understanding of existing systems. In the meantime, we will hopefully enrich the theory and our empirical understanding of these mechanisms, so that we will be equipped to handle these new challenges.

Summary of important notation:

π	strict priority ordering for all schools ($\pi = (\pi_1, ..., \pi_n)$)
π_s	strict priority ordering for school s
ψ^f	serial dictatorship with ordering f
φ^π	top trading cycles when priorities are π
m^{ψ^f}	matching produced from serial dictatorship with ordering f
m^{φ^π}	matching produced from top trading cycles when priorities are π
A_t	set of students in step t
A_t^s	set of students in step t, substep s
S_t	set of schools assigned to students in A_t
S_t^s	set of schools assigned to students in A_t^s
n_t	number of students in step t
n_t^s	number of students in step t, substep s
N_t	total number of students from step 1 to step t
A_t^*	set of unsatiated students in step t
Δ	set of priority skeletons
Δ_t	restriction of the set of priority skeletons to step t
δ	particular priority skeleton (element of Δ)
δ_i	component of a particular priority skeleton
$A(\delta_i)$	set of students in component of priority skeleton δ_i
$a_\ell(\delta_i)$	student in the last position in priority skeleton component δ_i
$\Pi(f)$	set of priorities for schools
$\Pi_t(f)$	set of priorities for schools in step t
$A_t^{1,+}$	set of satiated students in A_t^1
$A_t^{s,+}$	set of students in a higher position the priority skeleton among A_t^s
H_t^s	set of students not adjacent to last student in A_t^{s-1}
C_t	set of cycles in step t
C_t^s	set of cycles in step t, substep s
G_t	set of schools assigned in cycles in C_t
G_t^s	set of schools assigned in cycles in C_t^s
A_{C_t}	set of students assigned in cycles C_t
$A_{C_t^s}$	set of students assigned in cycles C_t^s
$A_{C_t^1}^+$	set of satiated students in $A_{C_t^1}$
$A_{C_t^s}^+$	set of students in a higher position in priority ordering among $A_{C_t^s}$

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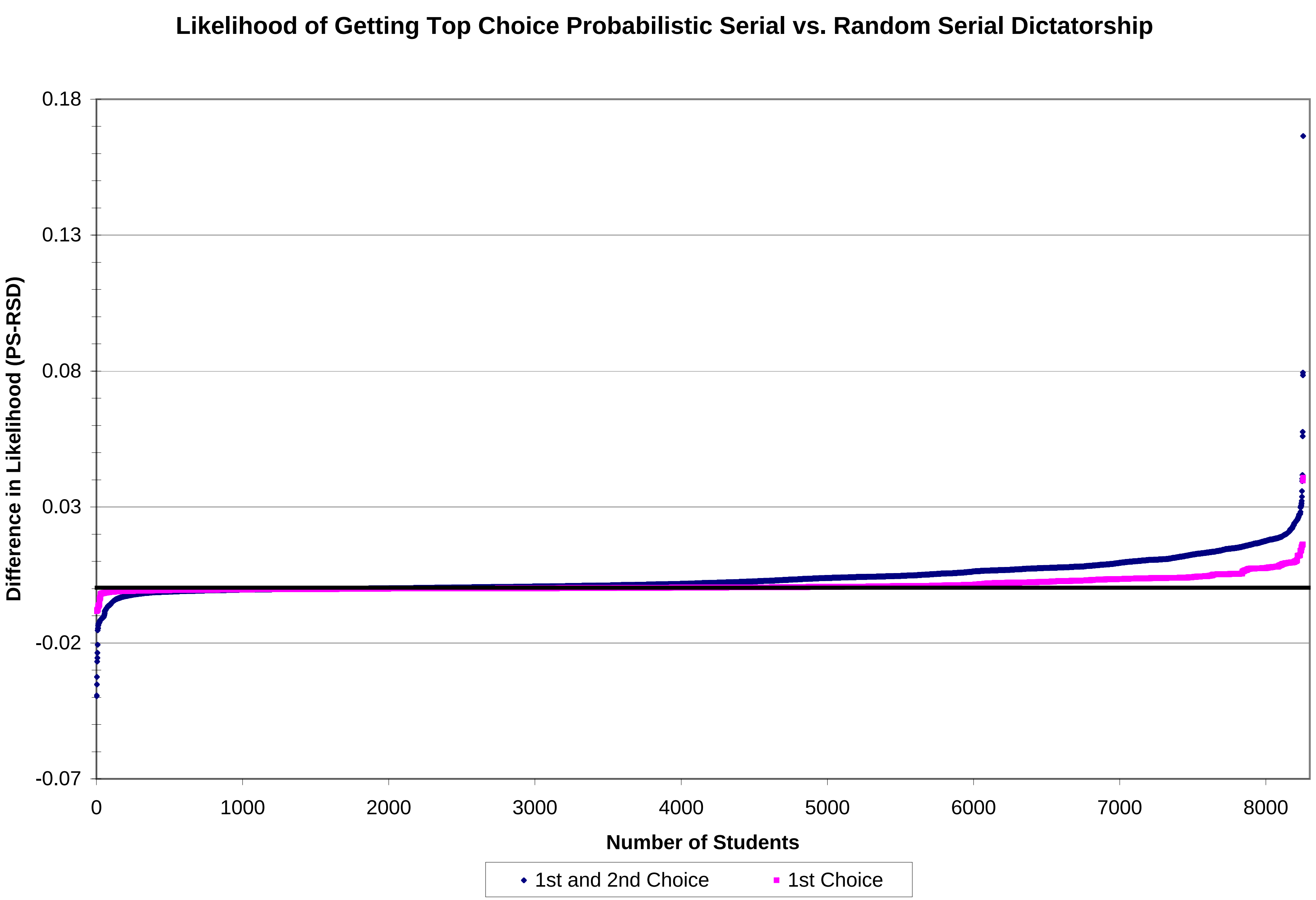

Likelihood of Getting Top Choice Probabilistic Serial vs. Random Serial Dictatorship
Difference in Likelihood (PS-RSD)
0.18
0.13
0.08
0.03
-0.02
-0.07
0
1000
2000
3000
4000
5000
6000
7000
8000
Number of Students
1st and 2nd Choice
1st Choice

Table 1— Comparing Mechanisms in the Supplementary Round in 2003-04[a]

Choice Received	Random Serial Dictatorship			Multiple Lottery Student-Proposing Deferred Acceptance	Probabilistic Serial
1	4999.05	4999.07	4999.07	3894.30	5015.66
2	1489.42	1489.41	1489.39	1887.26	1501.98
3	604.32	604.36	604.36	916.43	612.64
4	363.82	363.79	363.79	447.80	367.60
5	248.58	248.58	248.58	223.21	251.27
6	129.17	129.15	129.15	103.27	131.17
7	60.90	60.88	60.89	45.00	60.97
8	44.18	44.18	44.18	24.58	44.64
9	32.33	32.34	32.34	12.77	32.38
10	26.82	26.83	26.83	9.08	27.46
11	20.01	20.01	20.01	5.72	20.30
12	17.90	17.89	17.90	3.10	18.52
Unassigned	218.50	218.51	218.51	682.48	170.51
Iterations	250,000	500,000	1,000,000	1,000	-

[a]Constructed from data provided by the New York City Department of Education Office of High School Admissions. There are 8,255 students. Probabilistic Serial is calculated with the simultaneous eating algorithm with step size of 10^{-7}.

Table 2— Ordinal Efficiency versus Ex-Post Efficiency in Supplementary Round (2003-04)[a]

	Probabilistic Serial	Random Serial Dictatorship	No relation
Stochastic Dominance	4,126 50.0%	495 6.0%	3,634 44.0%
Likelihood of Receiving Top Choice Is Greater Under	67.6%	28.6%	
Likelihood of Receiving Top Two Choices is Greater Under	58.7%	12.7%	

[a]Constructed from data provided by the New York City Department of Education Office of High School Admissions. There are 8,255 students. Probabilistic Serial is calculated with the simultaneous eating algorithm with step size of 10^{-7}. The distribution of matchings from Random Serial Dictatorship (RSD) is approximated via a serial dictatorship with 1,000,000 different draws over the ordering of students. If a school program has N seats, it is treated as N separate programs, and students ranking the program rank N programs in a pre-specified order that is the same for all students.